On the fractal objects that arise from iteration of various zeta functions, L functions, and one modular form

David Rainford

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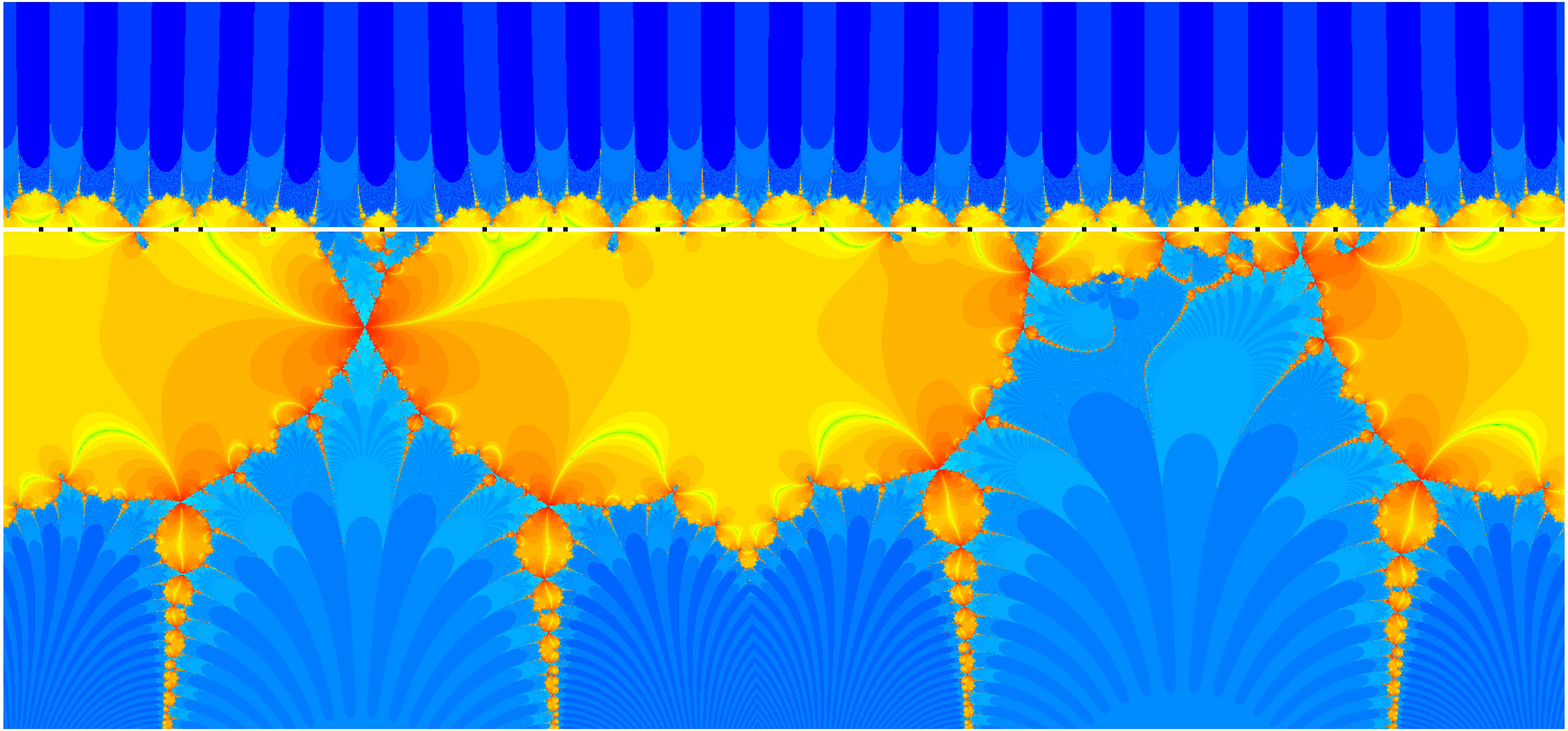
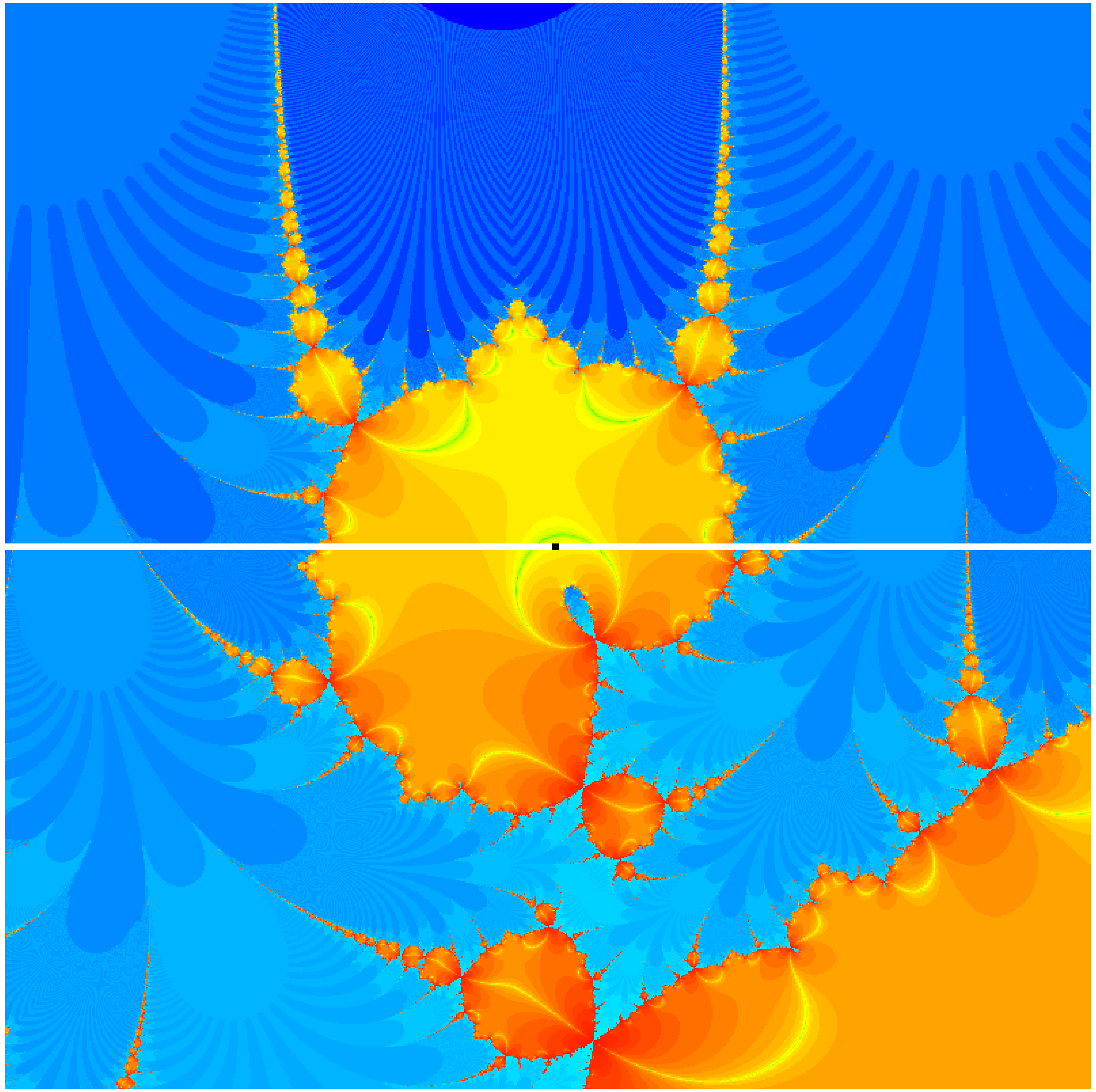
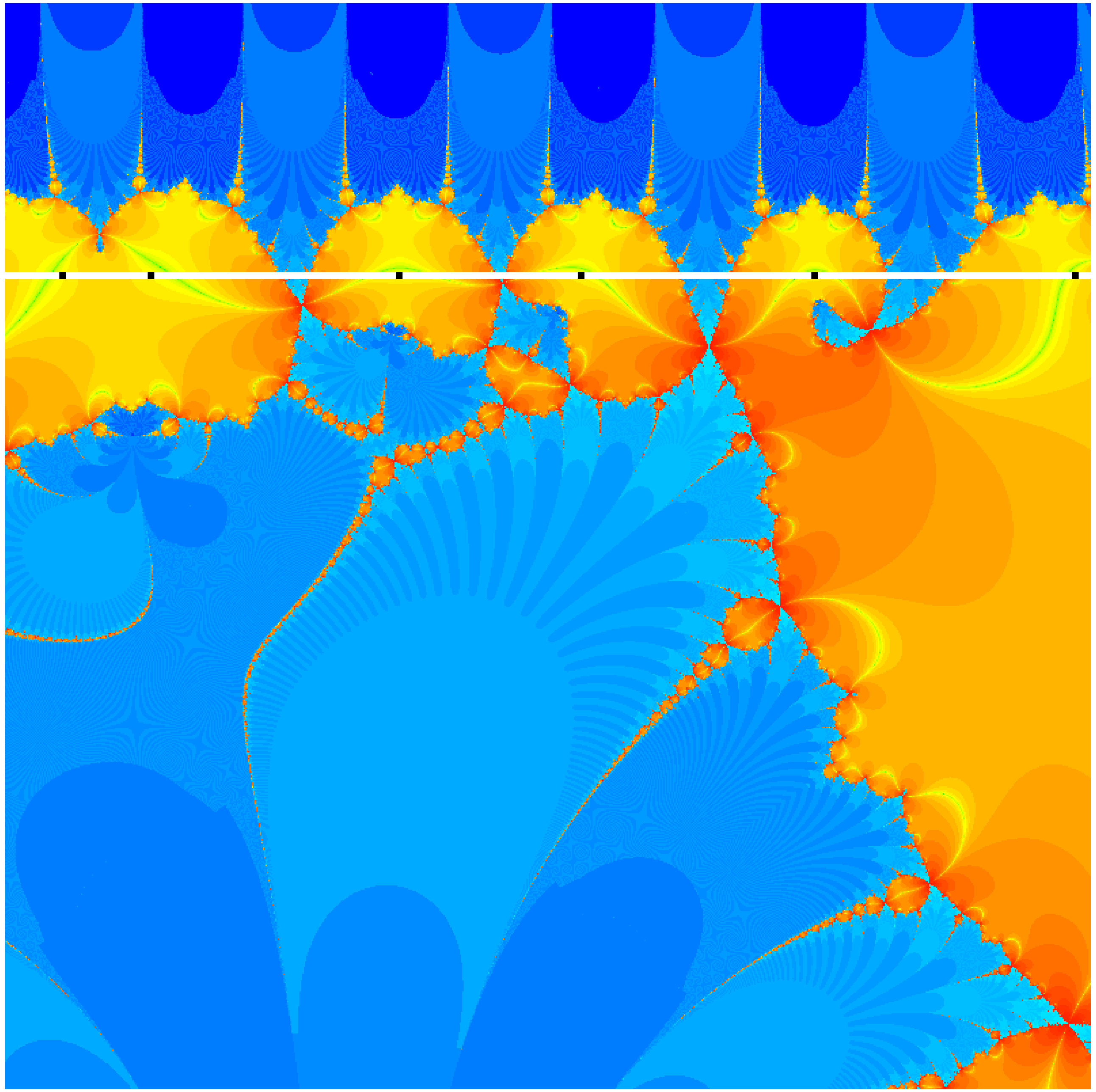




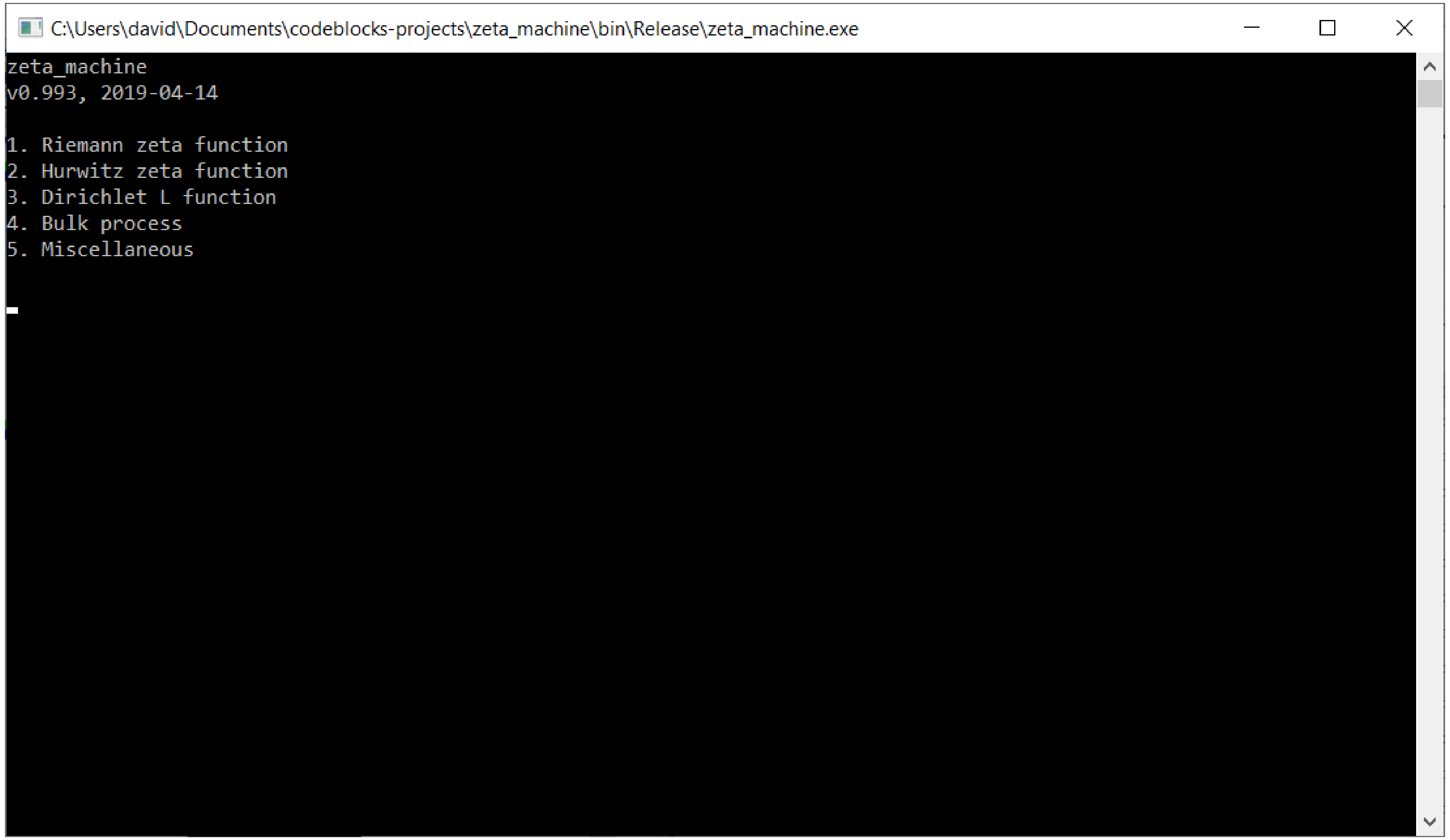


Like all pure mathematicians, [Apostol] was familiar with the Riemann Hypothesis, but he never saw himself as a serious contender.

'I had no idea how to approach it,' he said. 'I had read all the stuff and realized it was a very difficult problem. Some of us talked about it, and I remember once somebody said, "I had a dream one night - take zeta of zeta of s." Well, I never tried to push that.'

Dr Riemann's Zeros
Karl Sabbagh
Atlantic Books, 2003, p.77

zeta_machine



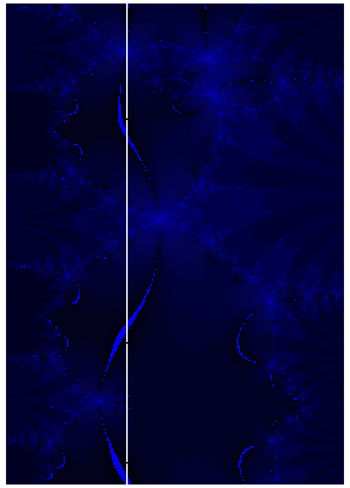
https://drive.google.com/open?id=1WIYC9Z6aN1mNGmPizHf3pfm7WRCoh7eo

zeta_machine

Inputs:
Real low (double)
Real high (double)
Imaginary low (double)
Imaginary high (double)
Resolution (pixels per unit, integer)
Optional: Hurwitz a (rational, double)
Optional: Dirichlet modulus (integer)
Optional: show critical line / non-trivial zeros (RZF only)

Outputs:

Data file 1 (24 bit Windows BMP)

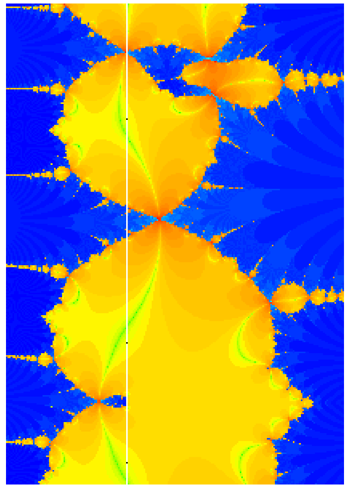


R byte = type (unsigned 8 bit char)
G byte
B byte
} iteration count (unsigned 16 bit short int)

Data file 2 (CSV)

iterations	type1	cid1	type2	cid2
0	0	0	0	511
1	0	0	0	511
2	0	1	0	511
3	1705	1	0	511
4	6120	14	0	511
5	8123	27	0	511
6	4886	41	0	511
7	4001	54	0	511
8	2721	68	0	511
9	1083	81	0	511
10	714	94	0	511
11	380	108	0	511
12	224	121	0	511

Image file (24 bit Windows BMP)



zeta_machine

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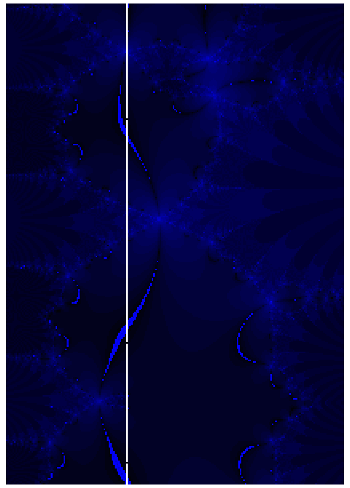

Data file 2 (CSV)

	A	B	C	D	E
1	iterations	type1	cid1	type2	cid2
2	0	0	0	0	511
3	1	0	0	0	511
4	2	0	1	0	511
5	3	1705	1	0	511
6	4	6120	14	0	511
7	5	8123	27	0	511
8	6	4886	41	0	511
9	7	4001	54	0	511
10	8	2721	68	0	511
11	9	1083	81	0	511
12	10	714	94	0	511
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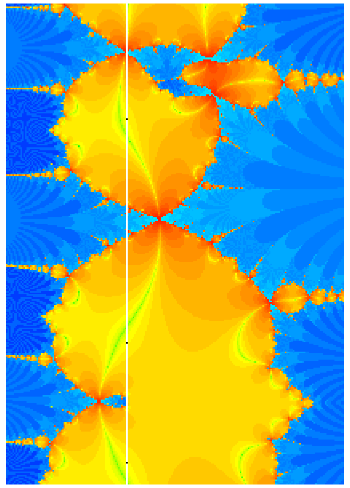
Image file (24 bit Windows BMP)



R byte = type (unsigned 8 bit char)
G byte
B byte $\Big\}$ iteration count (unsigned 16 bit short int)

Riemann zeta function

The Riemann zeta function $\zeta(s)$, for $s \in \mathbb{C}$, is defined as:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \Re(s) > 1 \tag{1}$$

The function has an analytic continuation to the whole complex plane, other than a simple pole at $s = 1$:

$$\zeta(1-s) = \frac{2\Gamma(s)}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right) \zeta(s) \qquad s \neq 1 \tag{2}$$

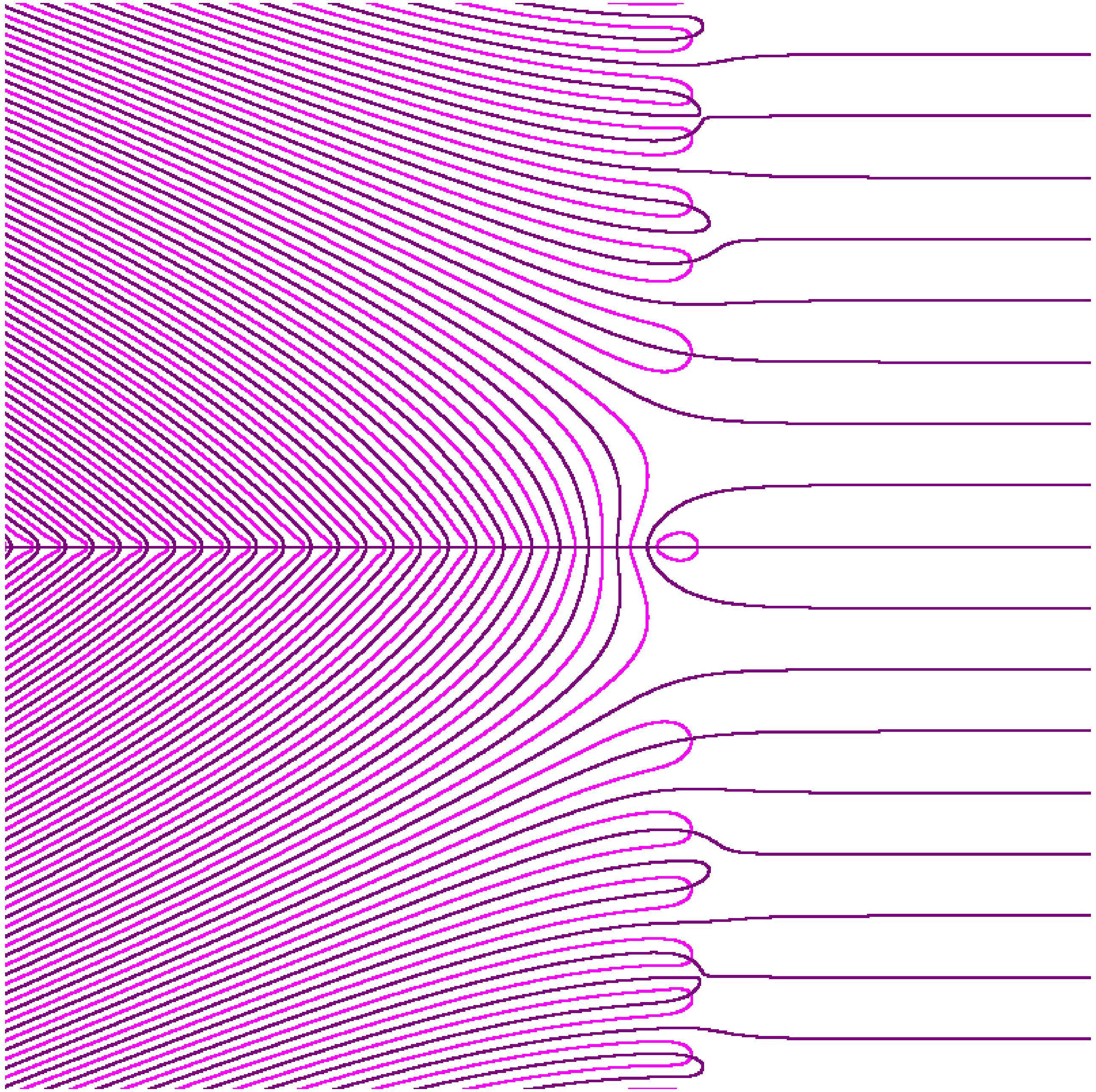
The Euler Maclaurin summation formula is the workhorse method for approximating the Riemann zeta function and is used in `zeta_machine`:

$$\zeta(s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + \frac{1}{2N^s} + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} \left(\prod_{j=0}^{2k-2} (s+j) \right) N^{1-s-2k} + R \tag{3}$$

The partial sum typically governs the rate at which the approximation proceeds because the number of terms, $N-1$, needs to be of the order of $|s|$. Some time can be saved by pre-calculating the Bernoulli terms in the second sum.

Riemann zeta function visualisations

$-50 \leq \Re(s) \leq 30, -40 \leq \Im(s) \leq 40, 10ppu$



$-50 \leq \Re(s) \leq 30, -40 \leq \Im(s) \leq 40, 10ppu$

Iteration

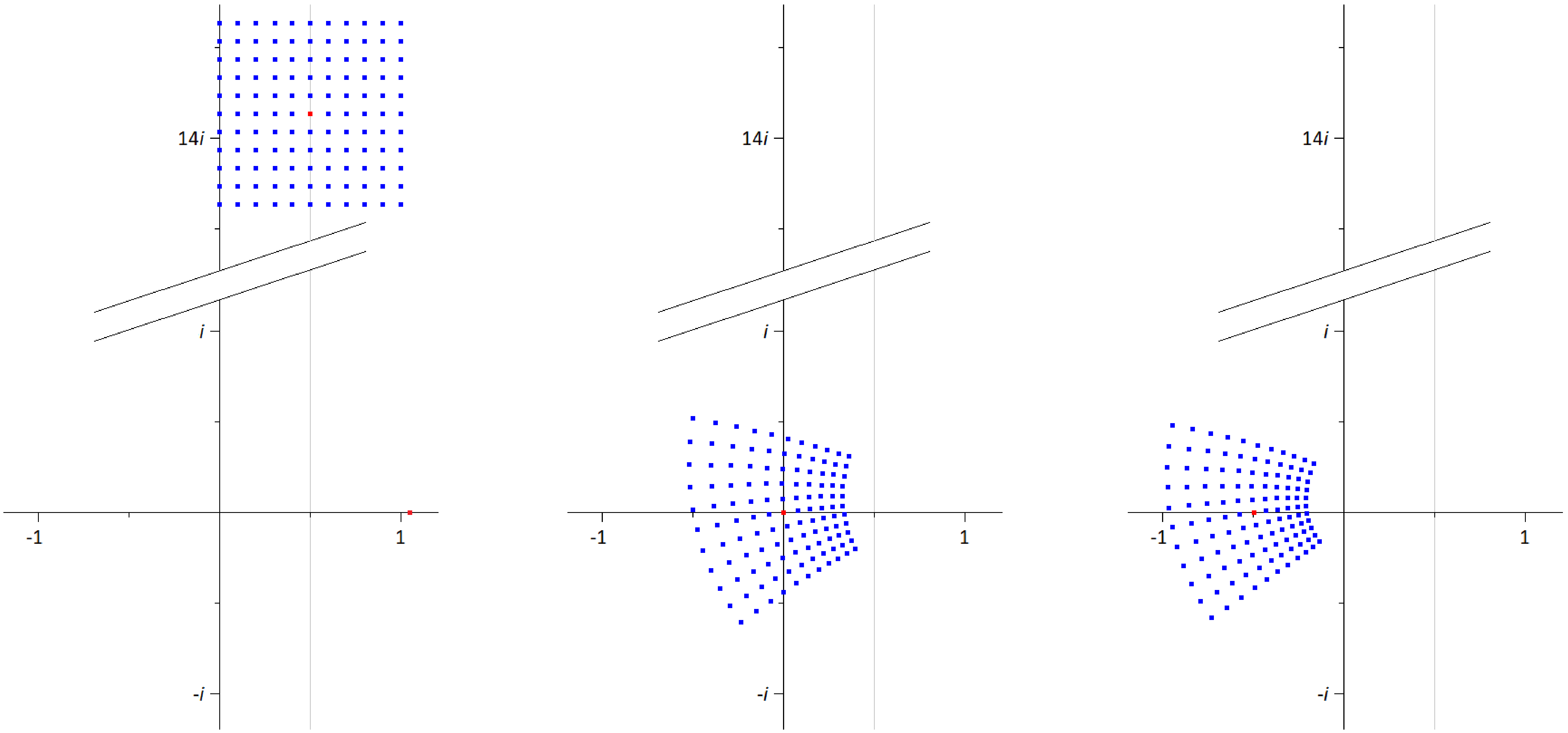
$$\zeta_0(s) = s$$

$$\zeta_{n+1}(s) = \zeta(\zeta_n(s))$$

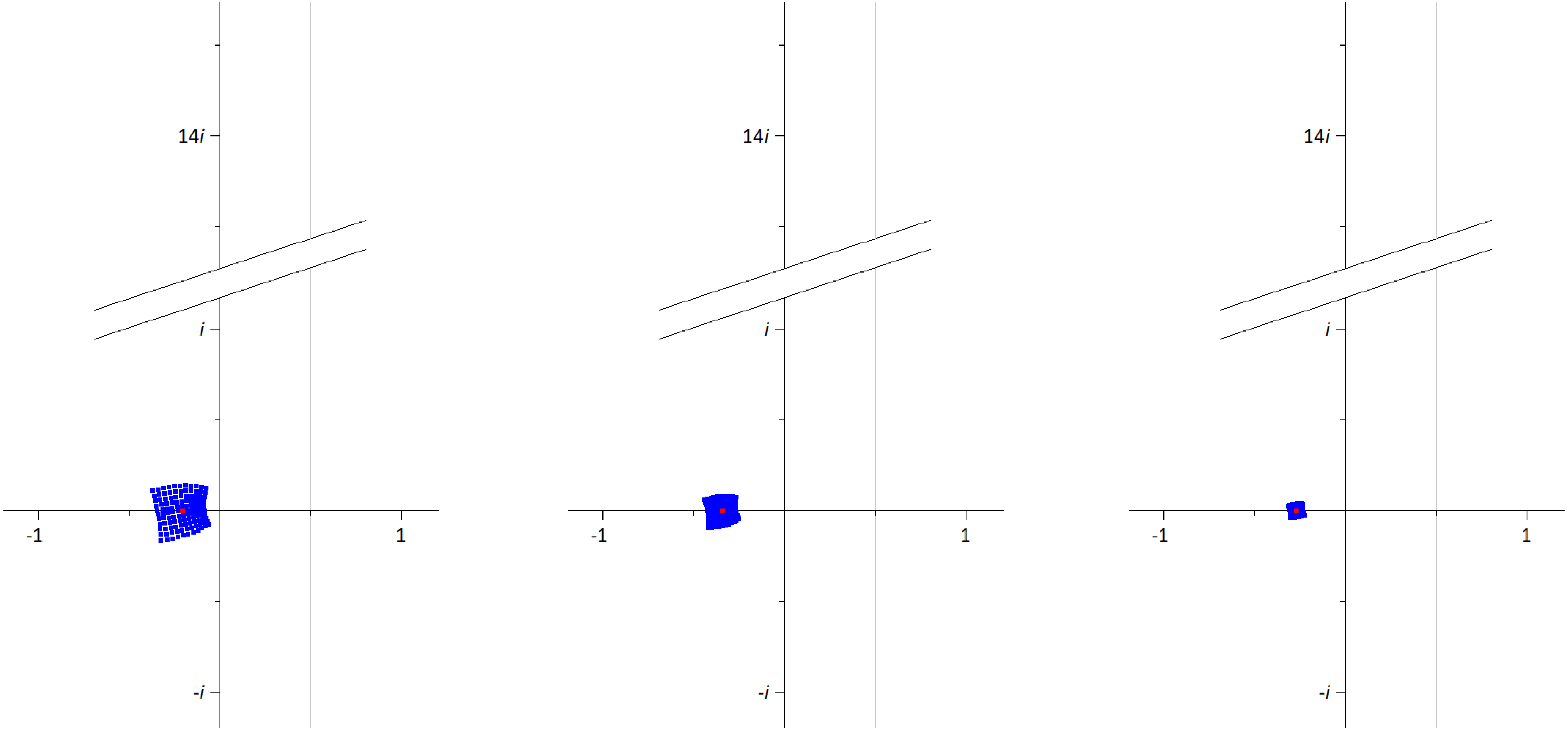
$$\text{Type 1: } \lim_{n\to\infty} \zeta_n(s) \to \infty$$

$$\text{Type 2: } \lim_{n\to\infty} \zeta_n(s) \to -0.29590500557521...$$

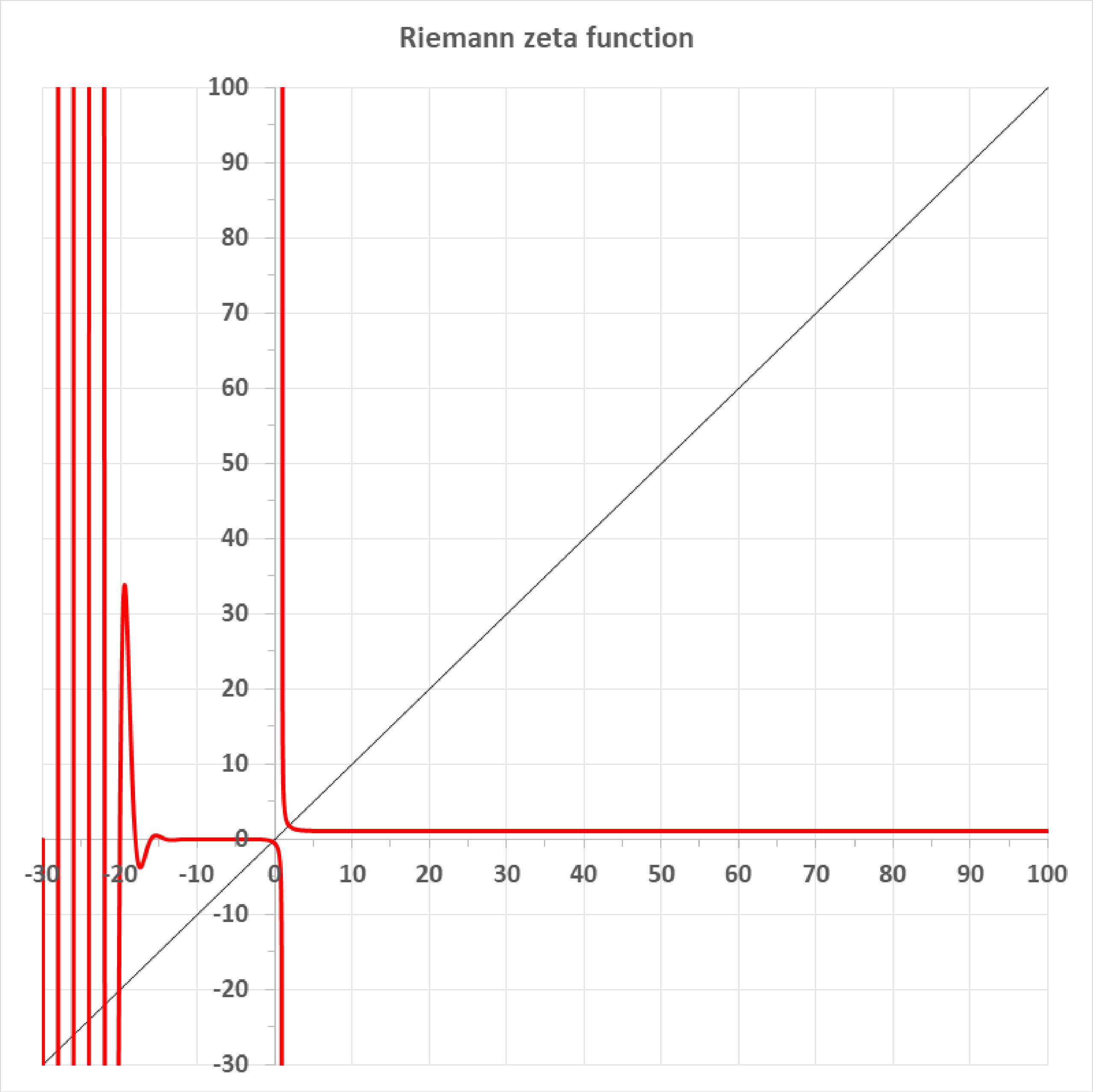
Iteration

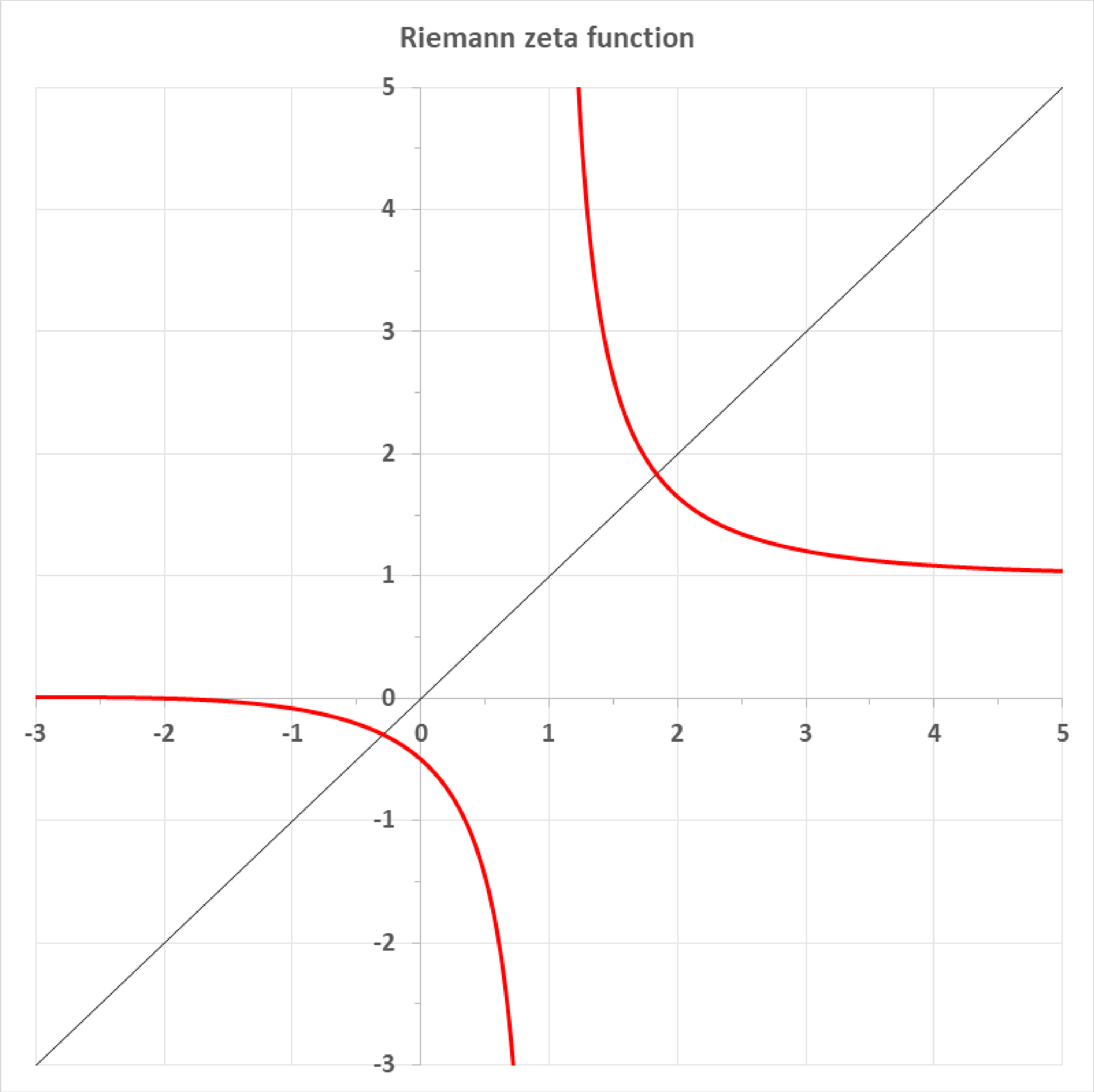


Iteration

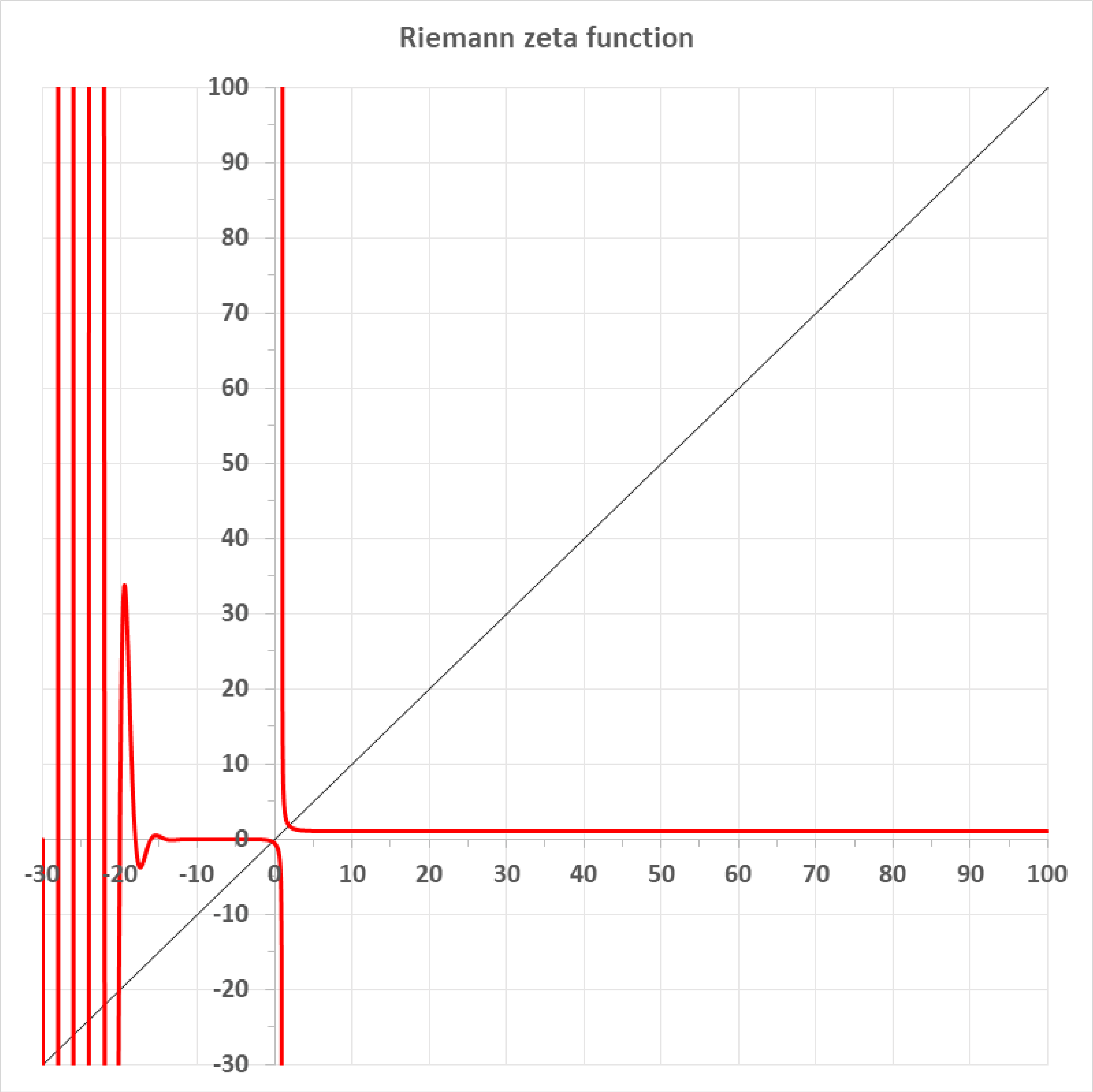


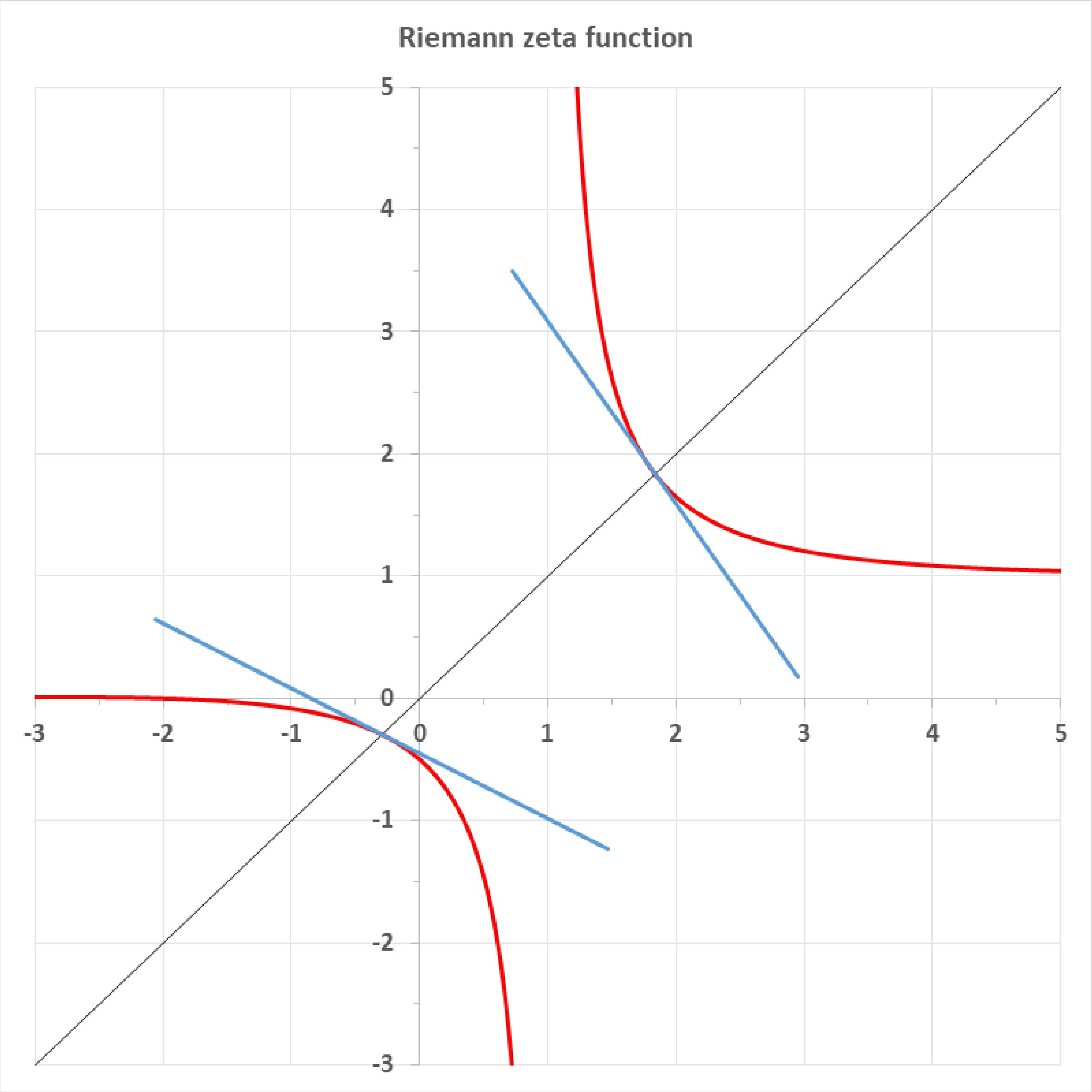
Fixed points



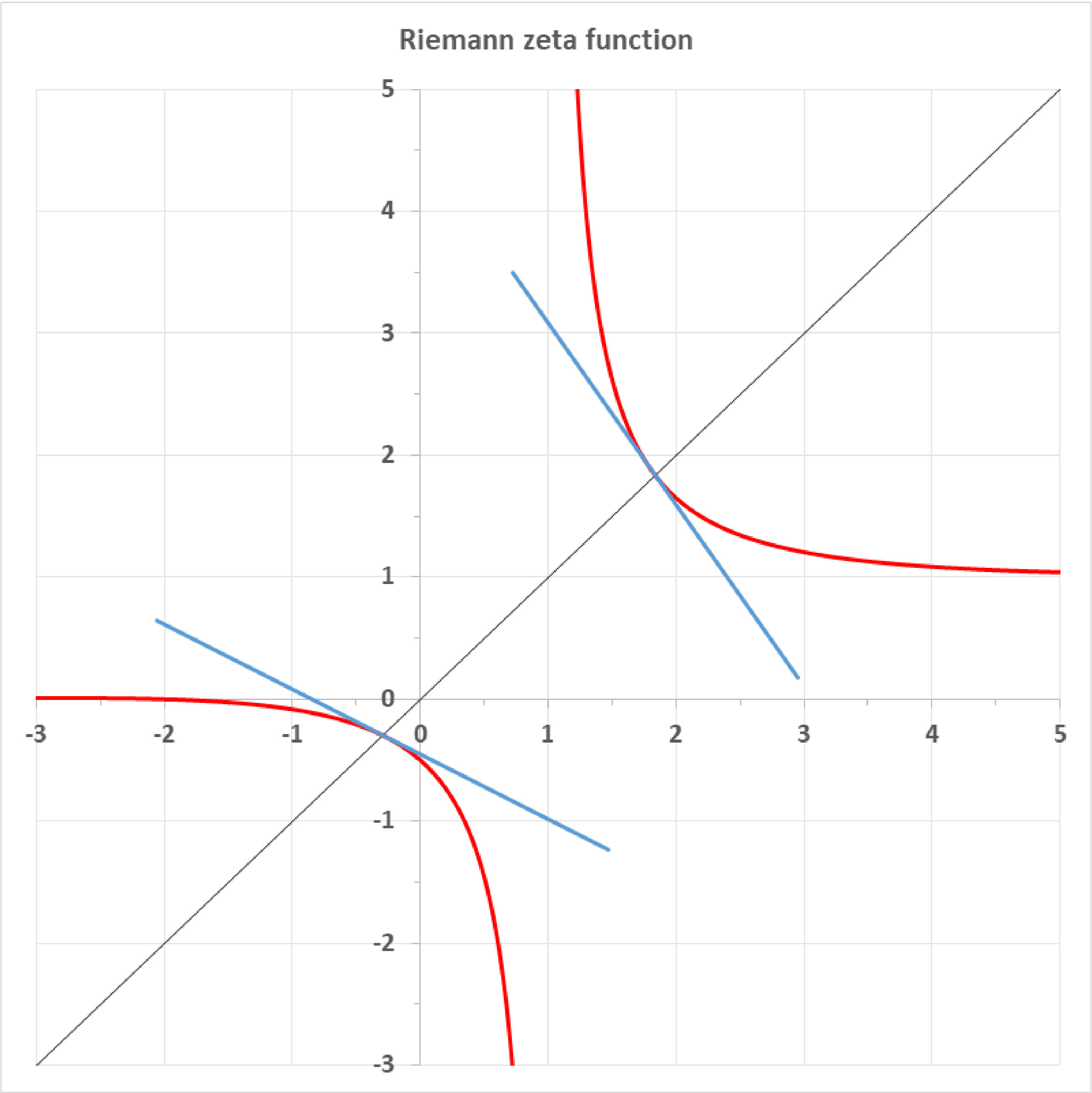


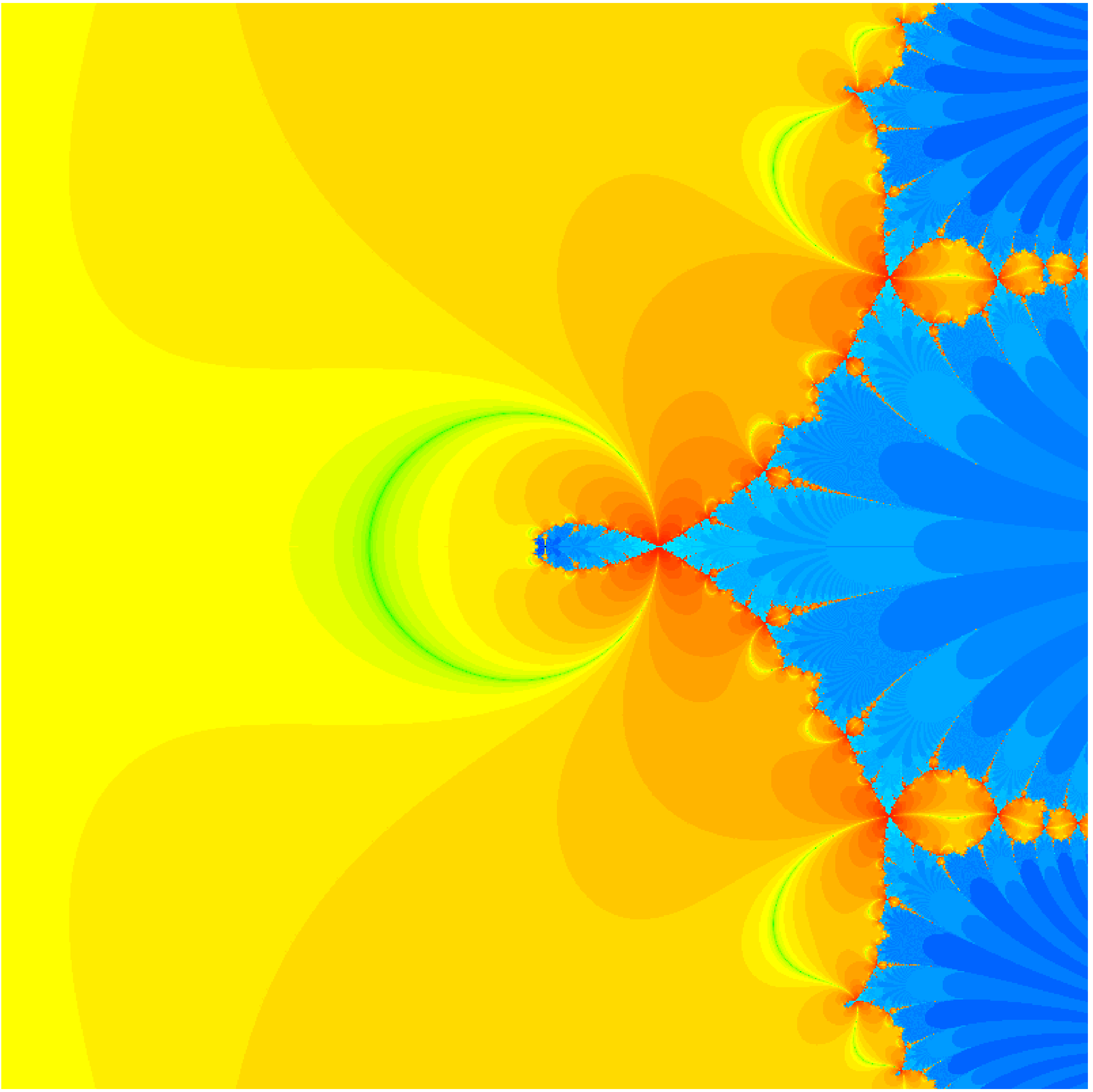
Fixed points





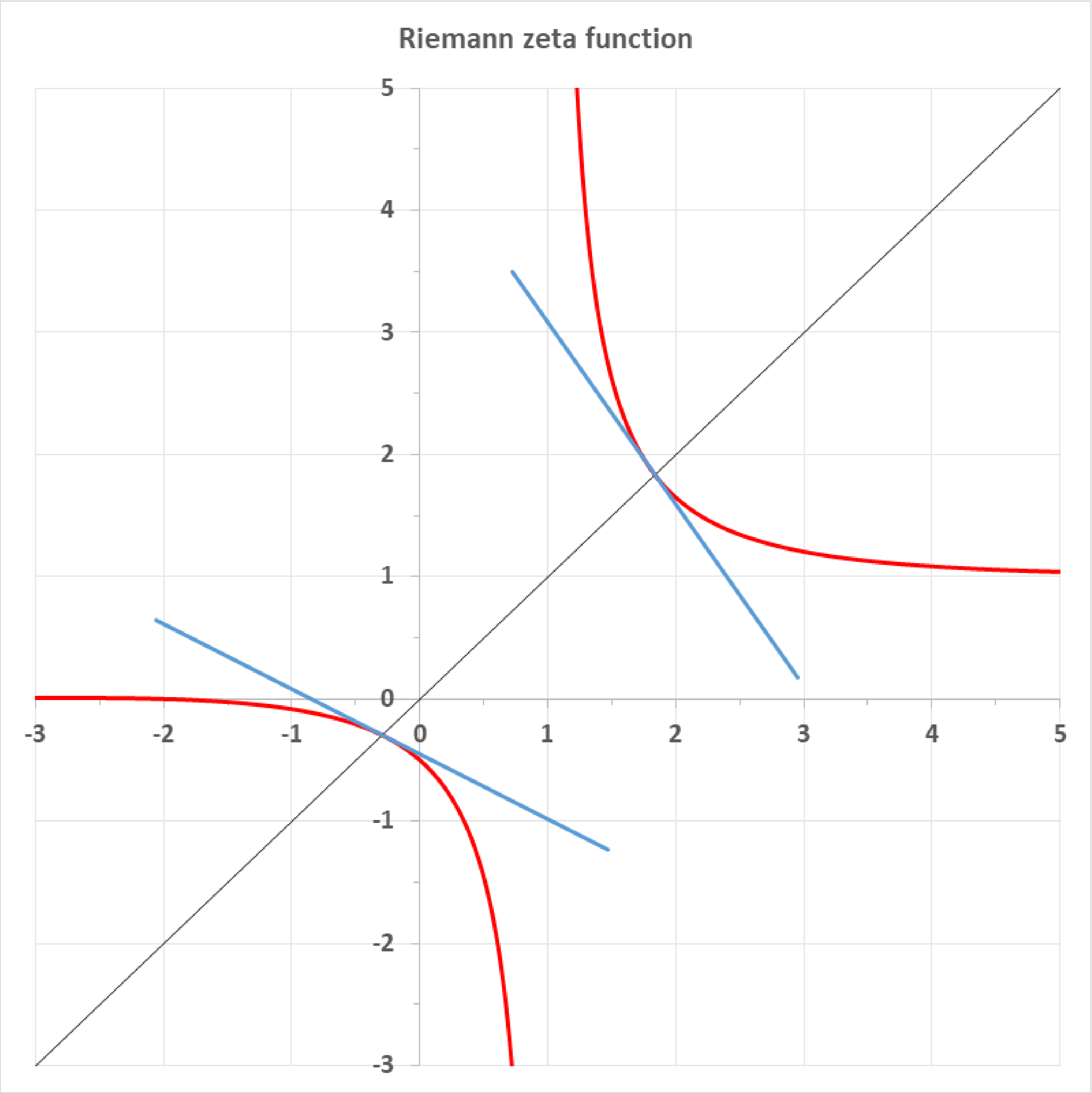
Fixed points

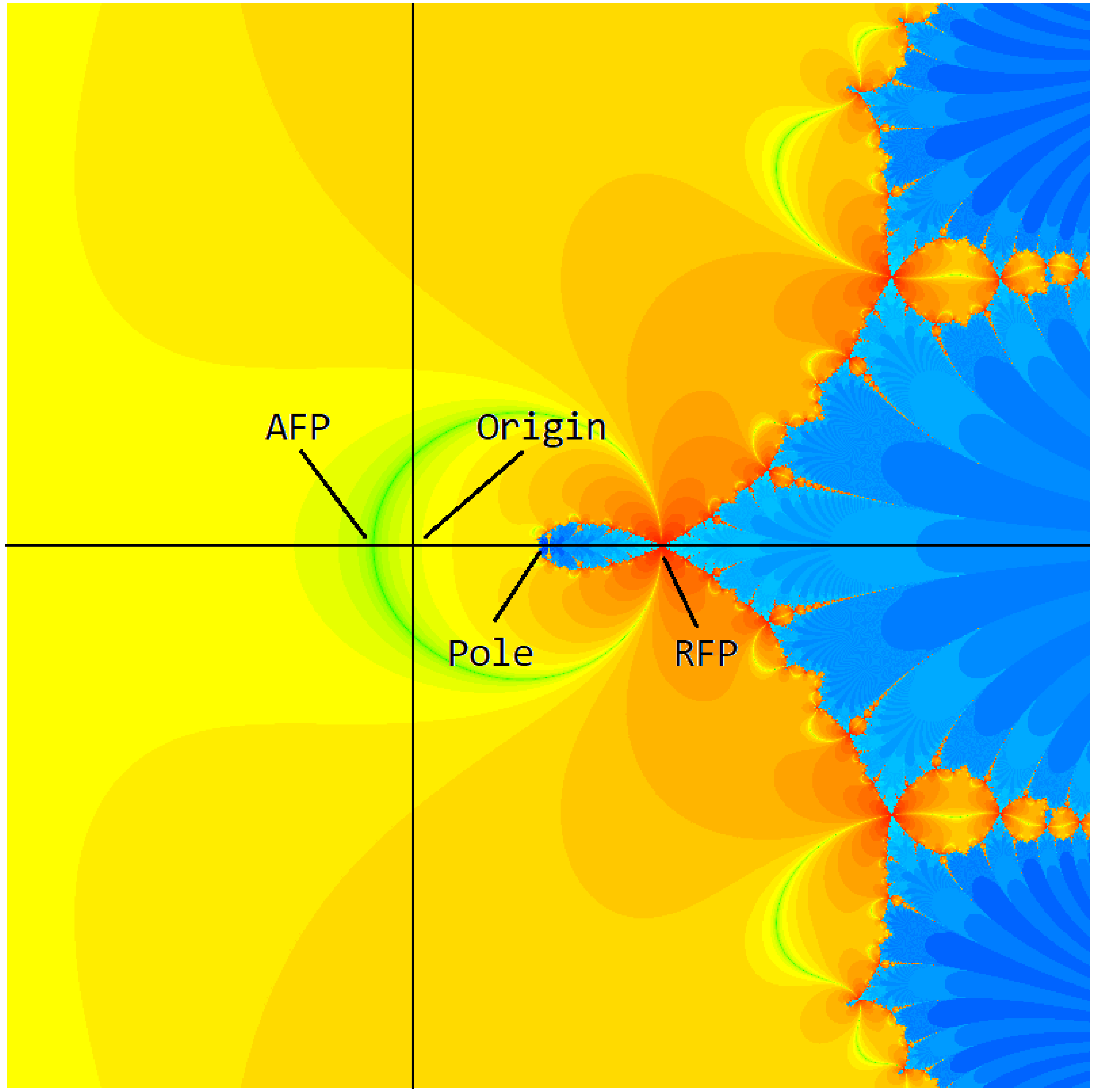




$-3 \leq \Re(s) \leq 5, -4 \leq \Im(s) \leq 4, 100ppu$

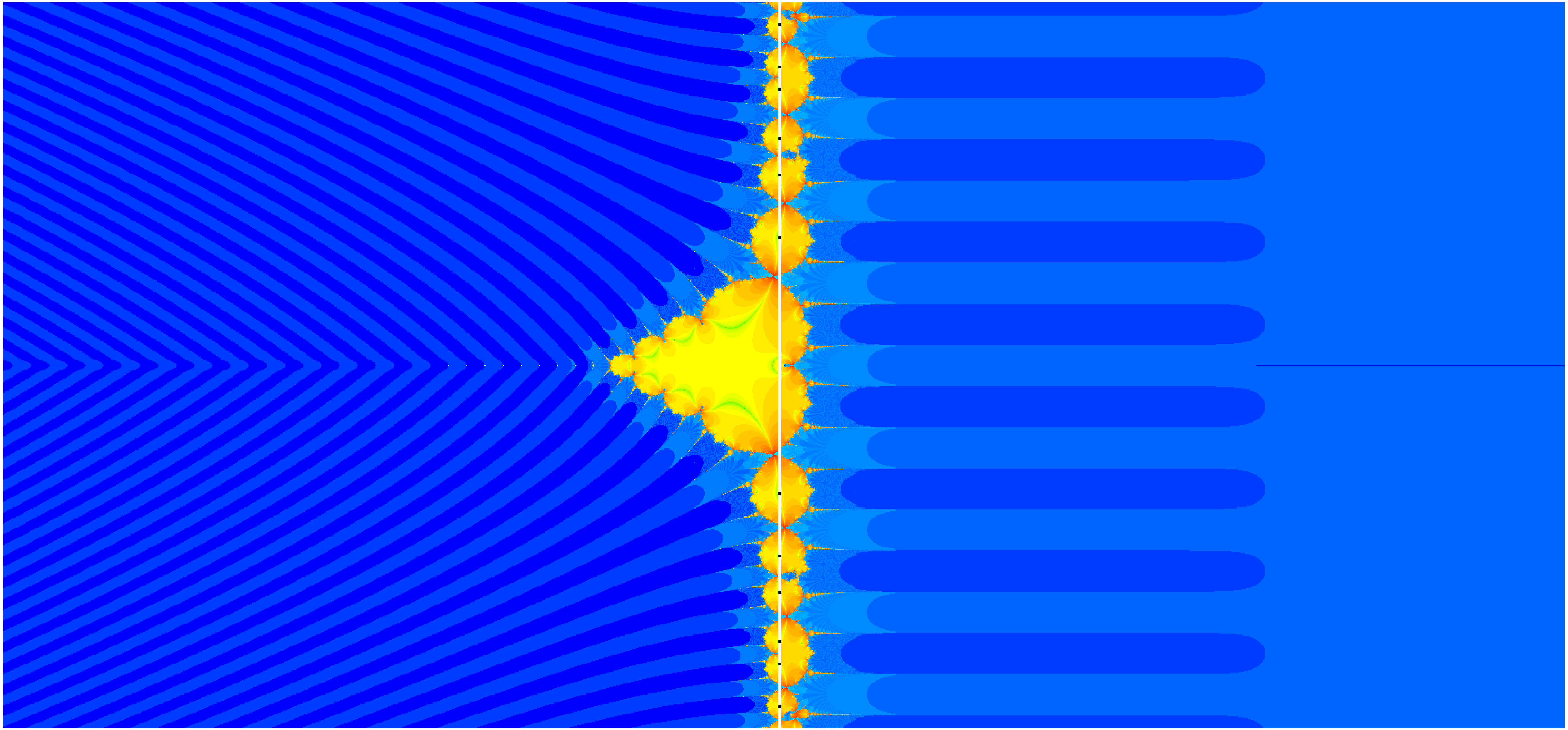
Fixed points





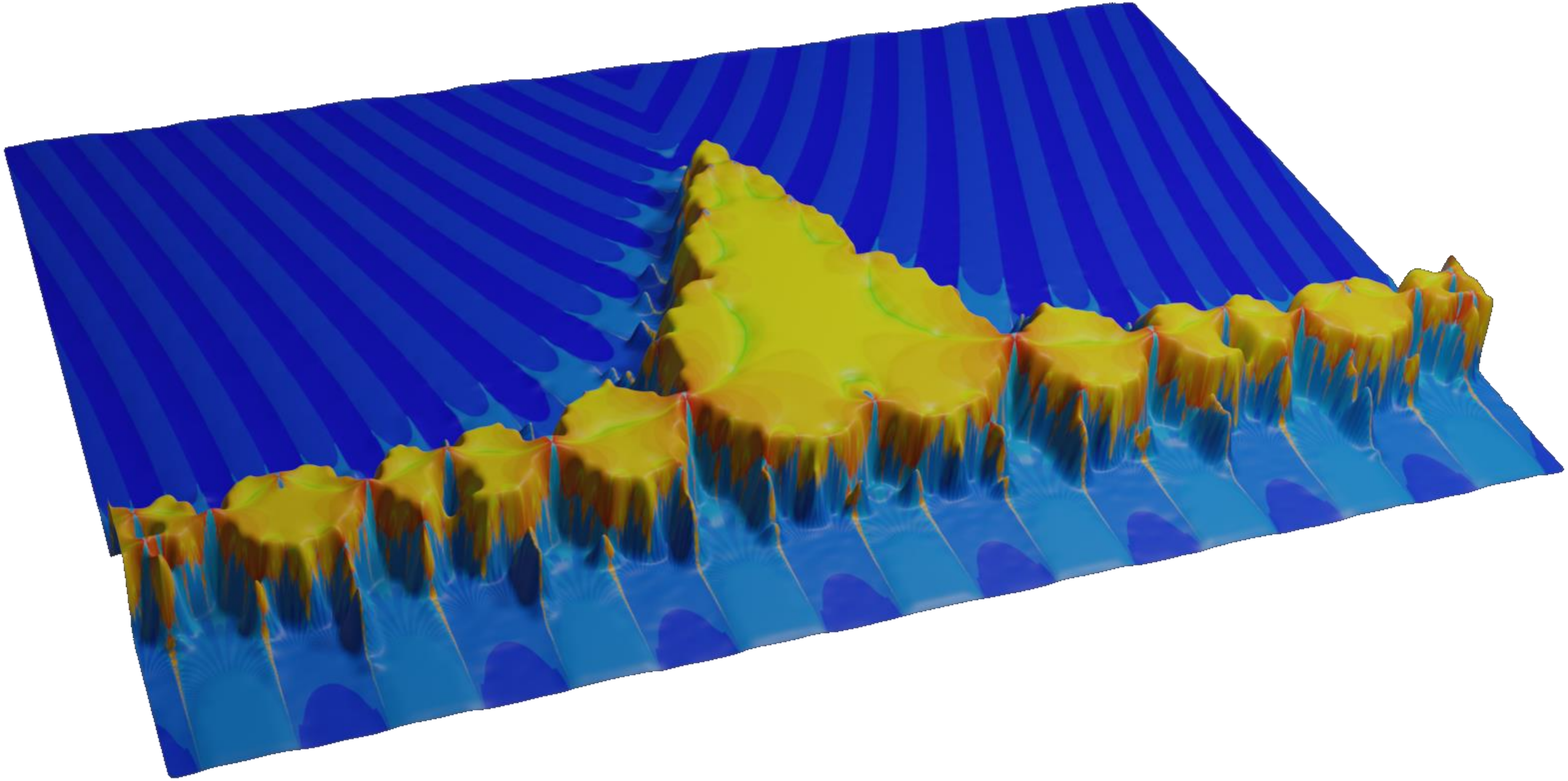
$-3 \leq \Re(s) \leq 5, -4 \leq \Im(s) \leq 4, 100ppu$

Riemann zeta function

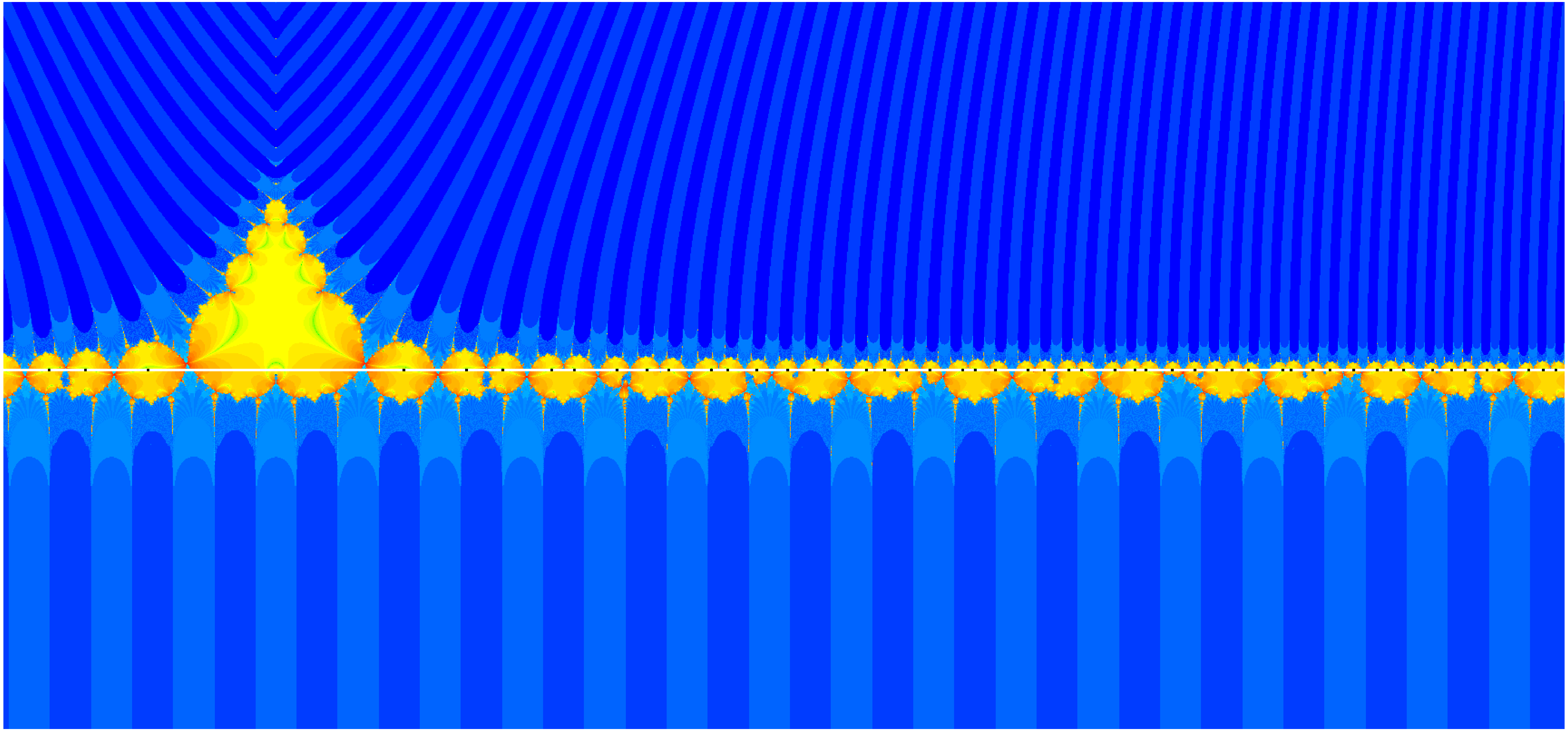


$-85 \leq \Re(s) \leq 87, -40 \leq \Im(s) \leq 40, 10ppu$

Riemann zeta function

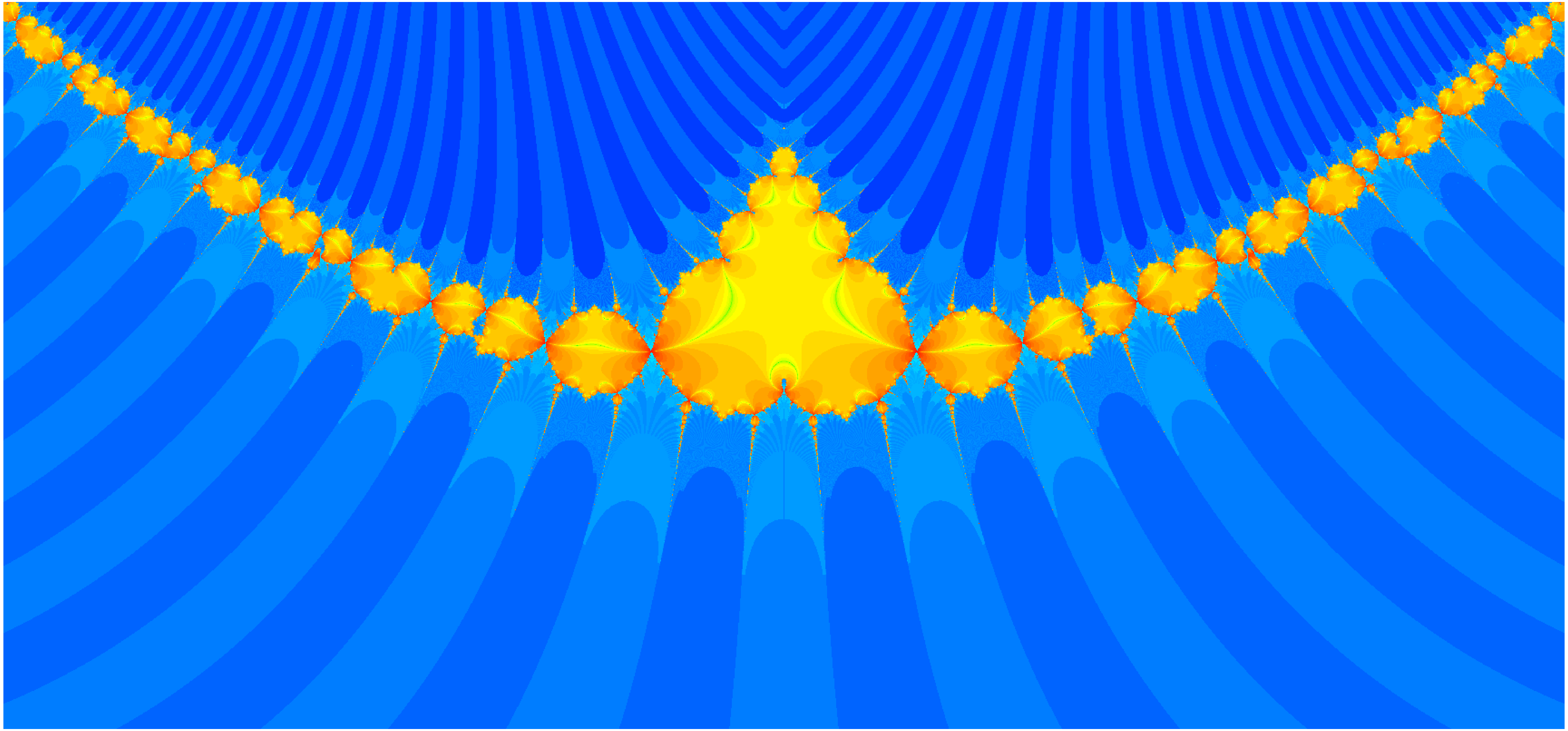


Riemann zeta function (10ppu)



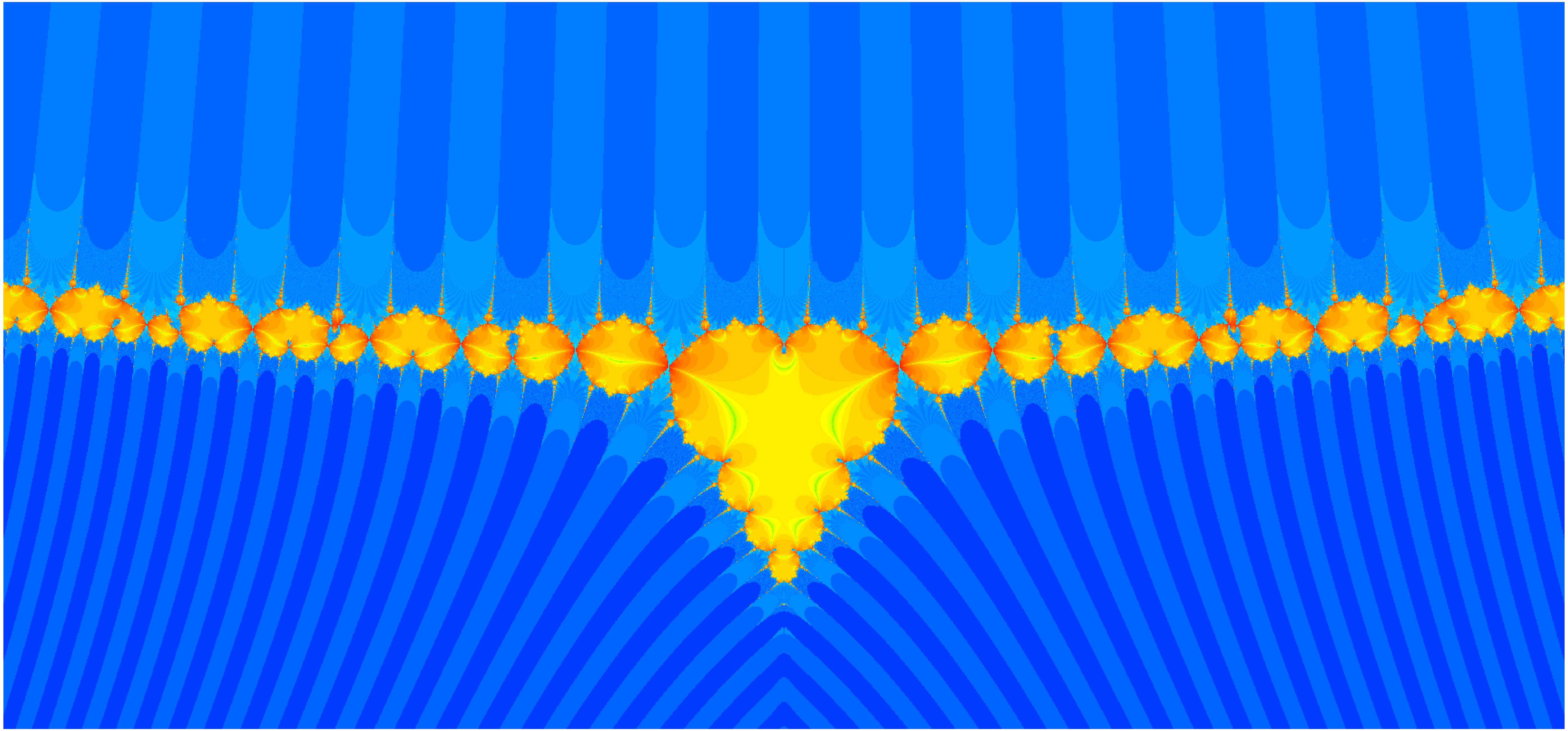
$-40 \leq \Re(s) \leq 40, -30 \leq \Im(s) \leq 142, 10ppu$

s = -20 (2,000ppu)



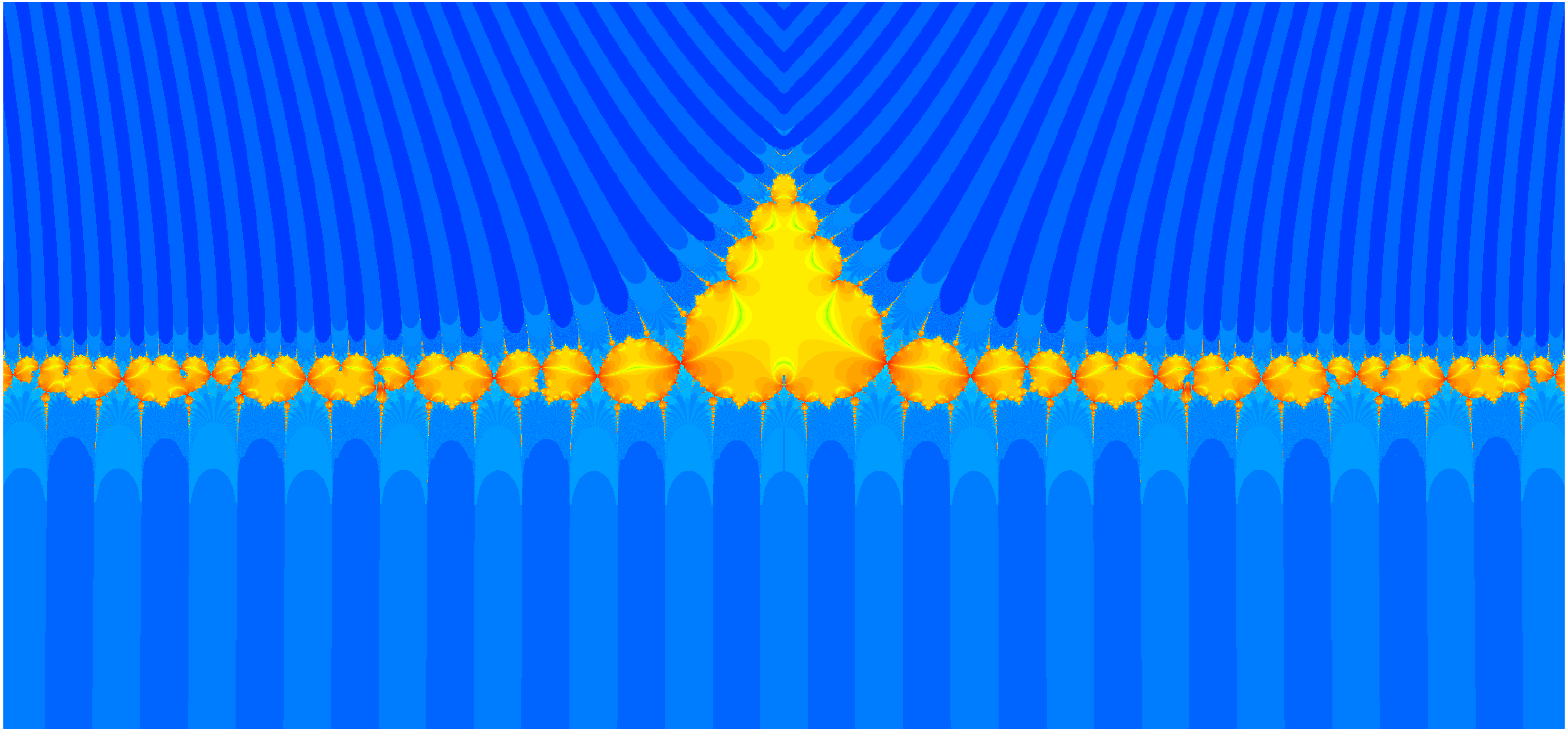
$-20.2 \leq \Re(s) \leq -19.8, -0.43 \leq \Im(s) \leq 0.43, 2000ppu$

s = -22 (20,000ppu)



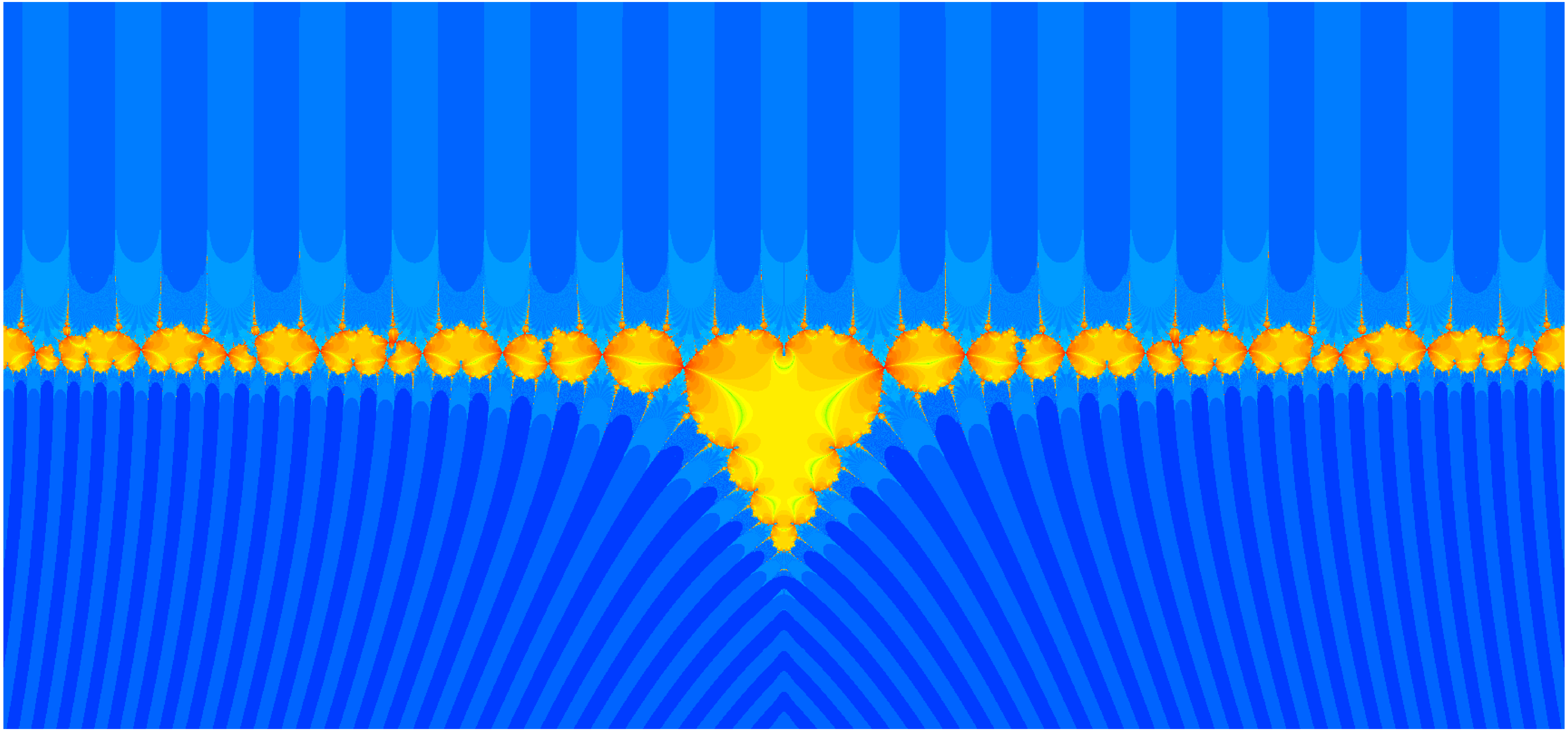
$-22.02 \leq \Re(s) \leq -21.98, -0.043 \leq \Im(s) \leq 0.043, 20000ppu$

s = -24 (250,000ppu)



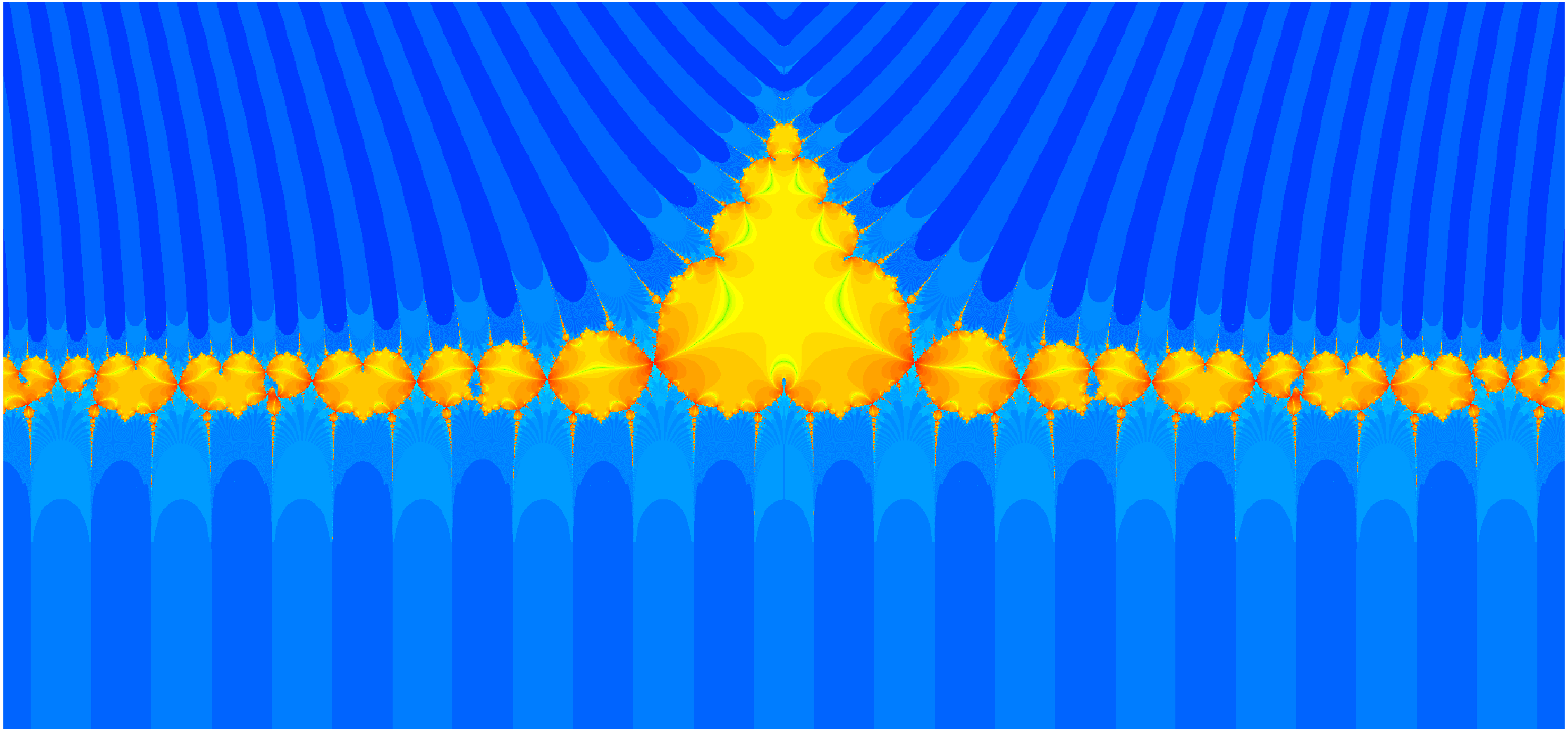
$-24.0016 \leq \Re(s) \leq -23.9984, -0.00344 \leq \Im(s) \leq 0.00344, 250000ppu$

s = -26 (4,000,000ppu)



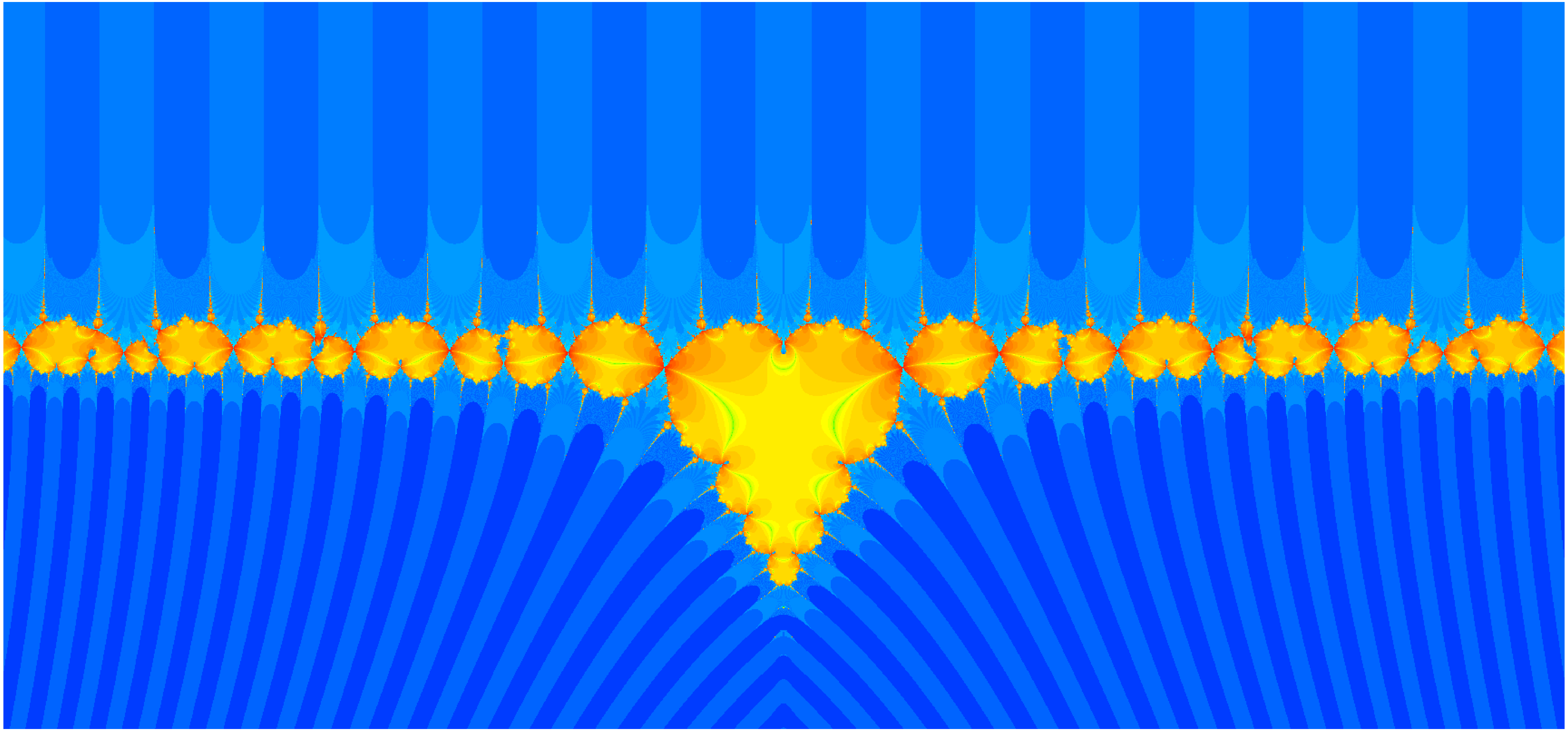
$-26.0001 \leq \Re(s) \leq -25.9999, -0.000215 \leq \Im(s) \leq 0.000215, 4000000ppu$

s = -28 (100,000,000ppu)



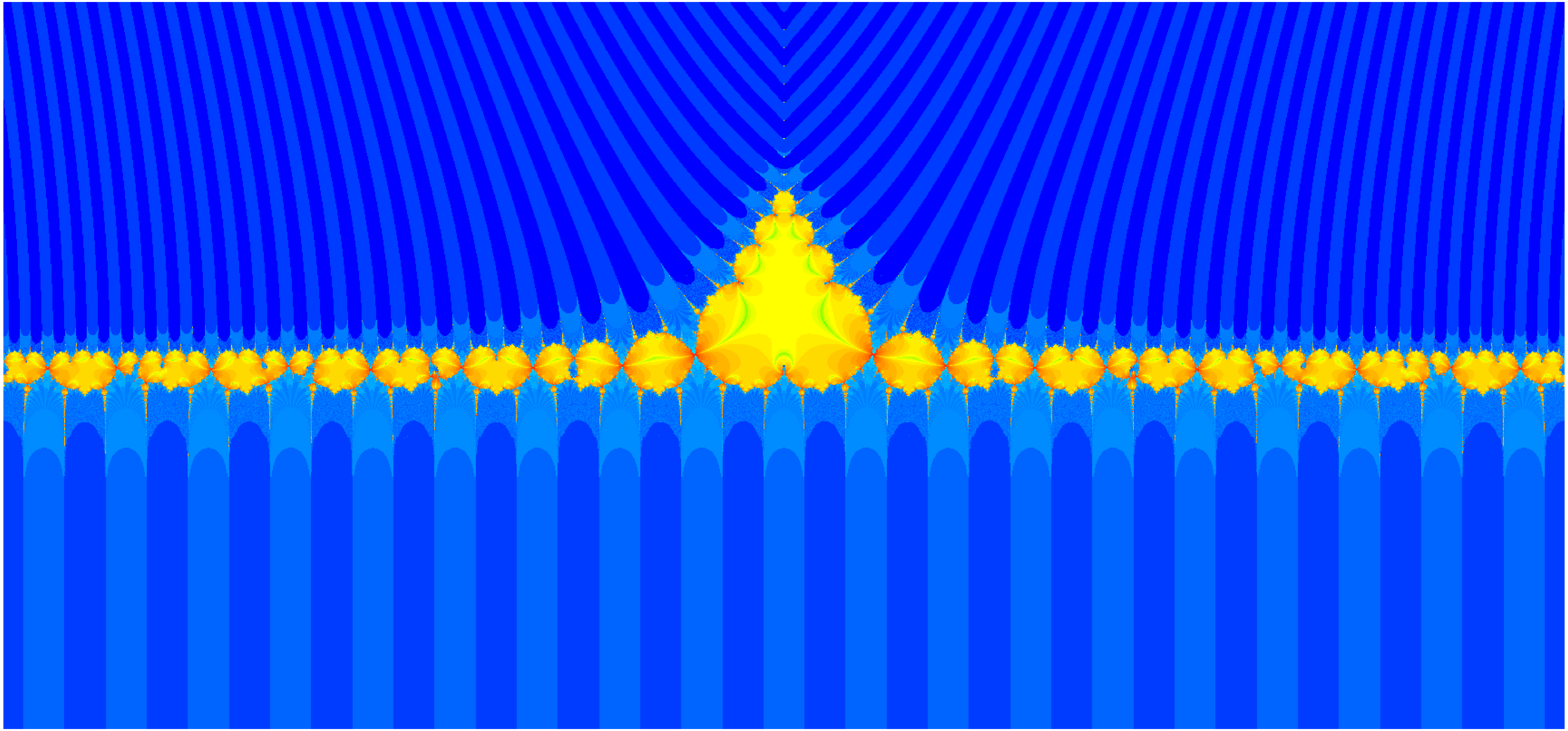
$-28.000004 \leq \Re(s) \leq -27.999996, -0.0000086 \leq \Im(s) \leq 0.0000086, 10000000ppu$

s = -30 (2,000,000,000ppu)



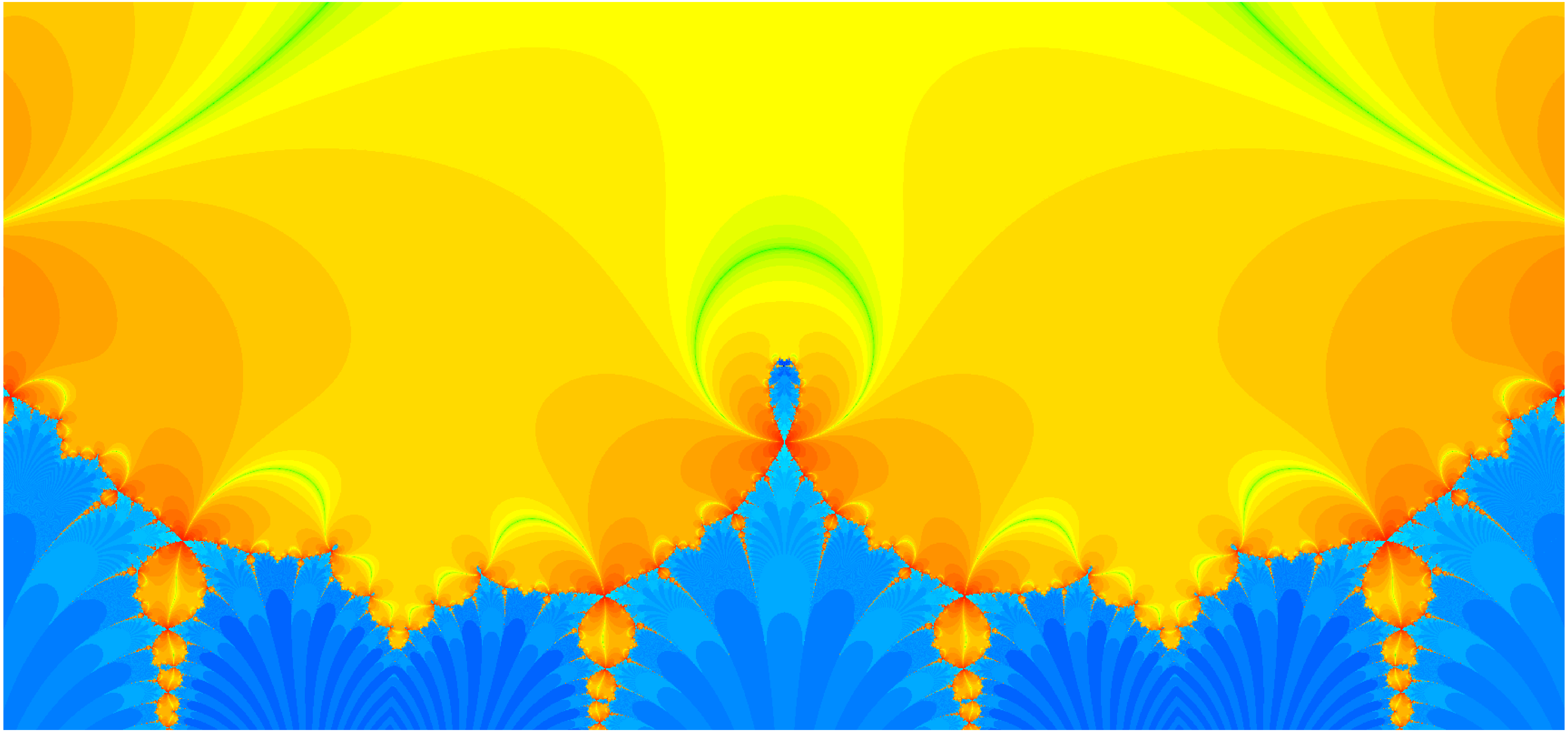
$-30.0000002 \leq \Re(s) \leq -29.9999998, -0.00000043 \leq \Im(s) \leq 0.00000043, 2000000000ppu$

Pole (10ppu)



$-39 \leq \Re(s) \leq 41, -86 \leq \Im(s) \leq 86, 10ppu$

Pole (100ppu)

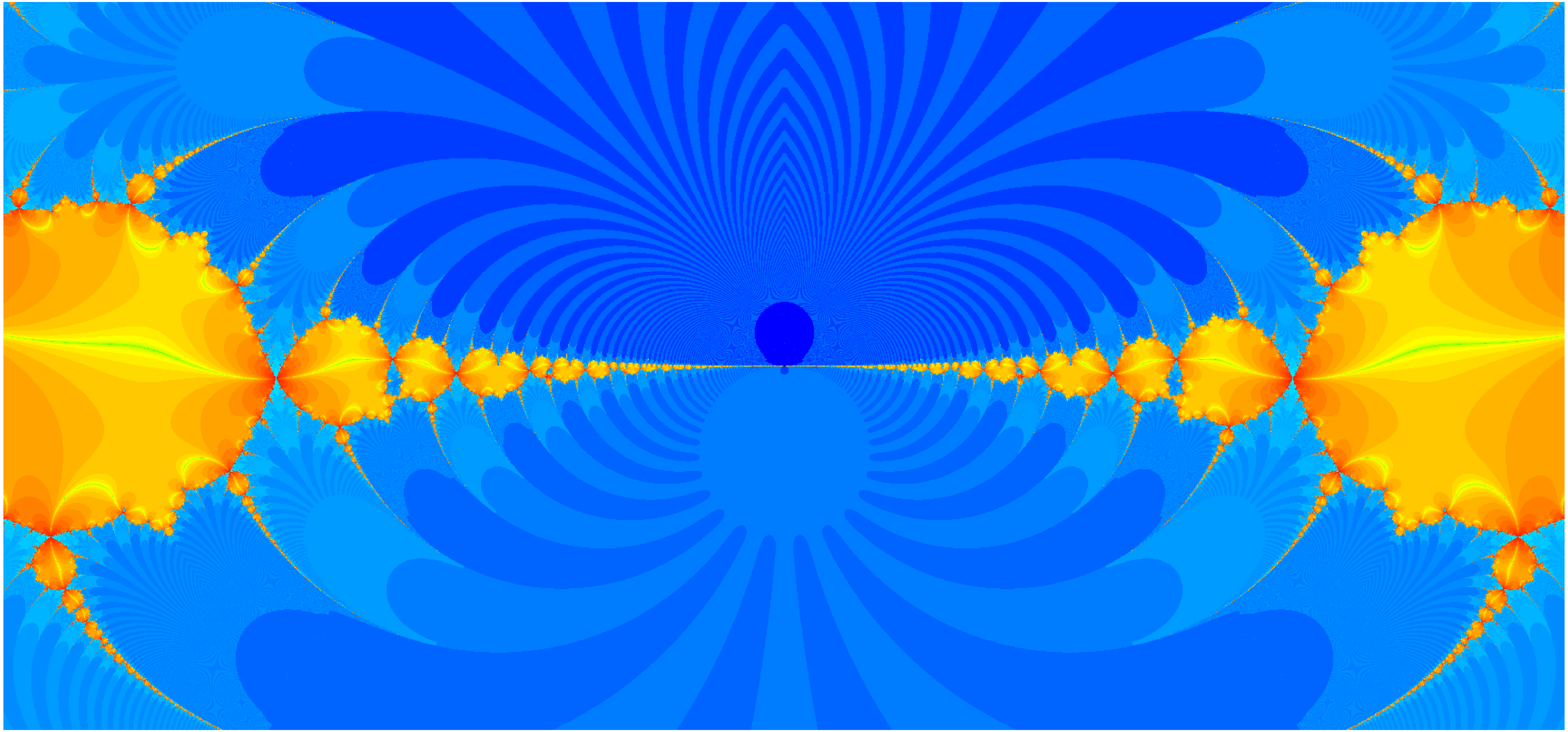


$-3 \leq \Re(s) \leq 5, -8.6 \leq \Im(s) \leq 8.6, 100ppu$

Pole (1,000ppu)

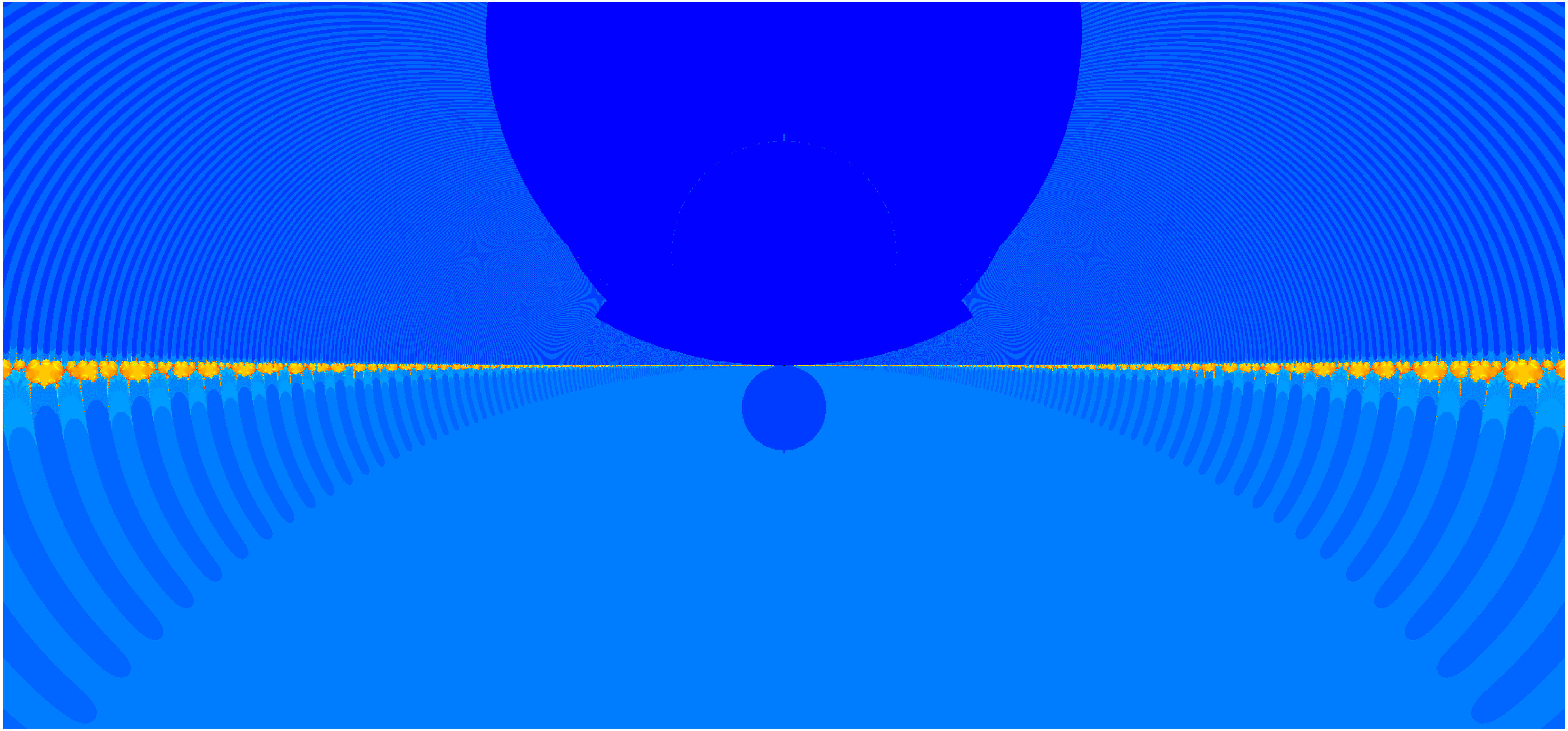
$-0.6 \leq \Re(s) \leq 1.4, -0.86 \leq \Im(s) \leq 0.86, 1000ppu$

Pole (10,000ppu)



$0.96 \leq \Re(s) \leq 1.04, -0.086 \leq \Im(s) \leq 0.086, 10000ppu$

Pole (100,000ppu)



$0.996 \leq \Re(s) \leq 1.004, -0.0086 \leq \Im(s) \leq 0.0086, 100000ppu$

Riemann zeta function

Observations / assertions:

1. Each non-trivial zero is associated with a “segment”, with exactly one non-trivial zero per segment

2. The fractal surface consists of “clumps”, and each clump has one or more non-trivial zeros

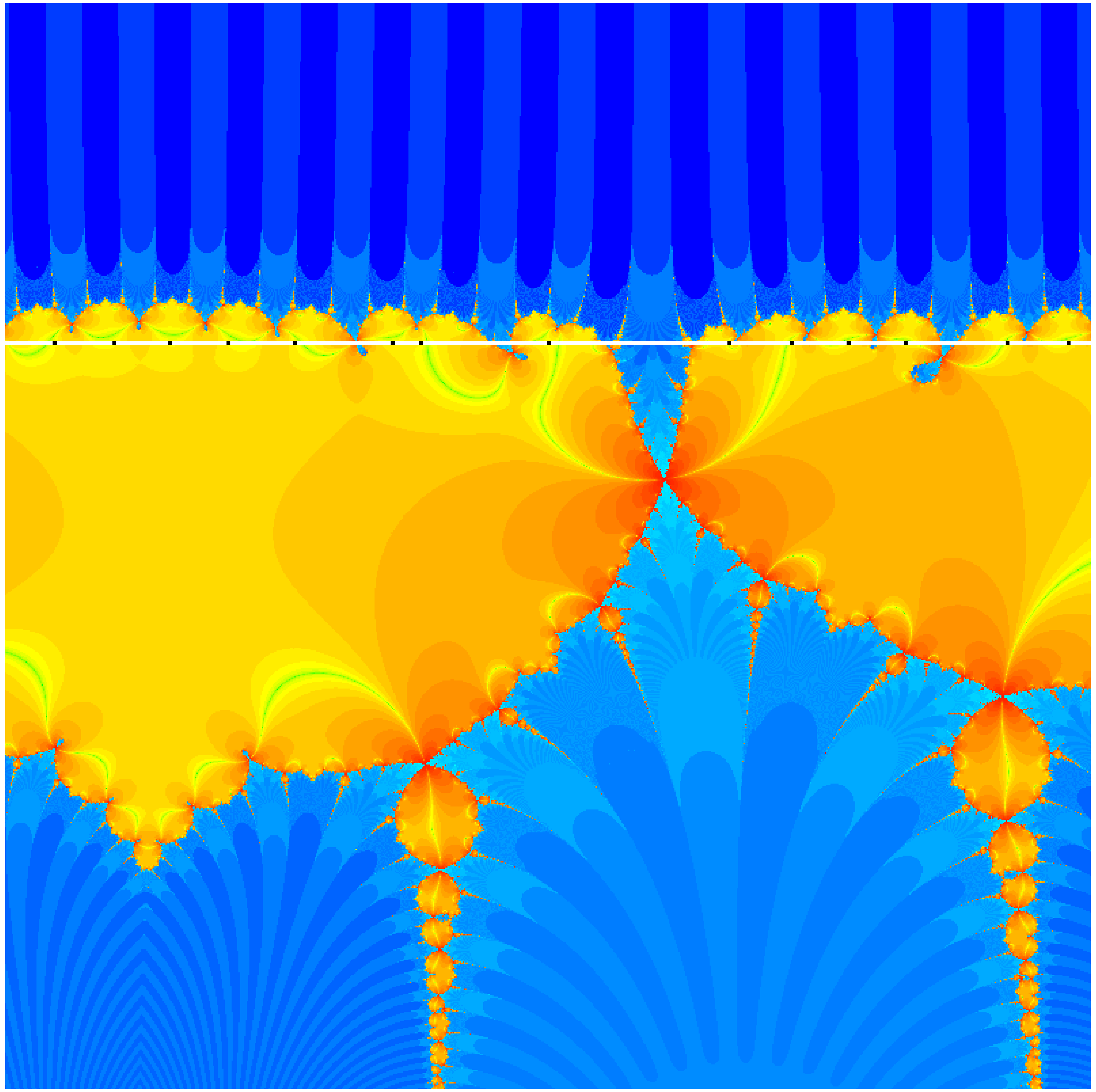
3. The neighbourhood of each non-trivial zero is a scaled and distorted reproduction of the whole of the main bulb, with the non-trivial zero occupying a position corresponding to the origin

4. In particular, the reproductions of the neighbourhood of the main bulb closest to the pole dominate the lower regions of the clumps

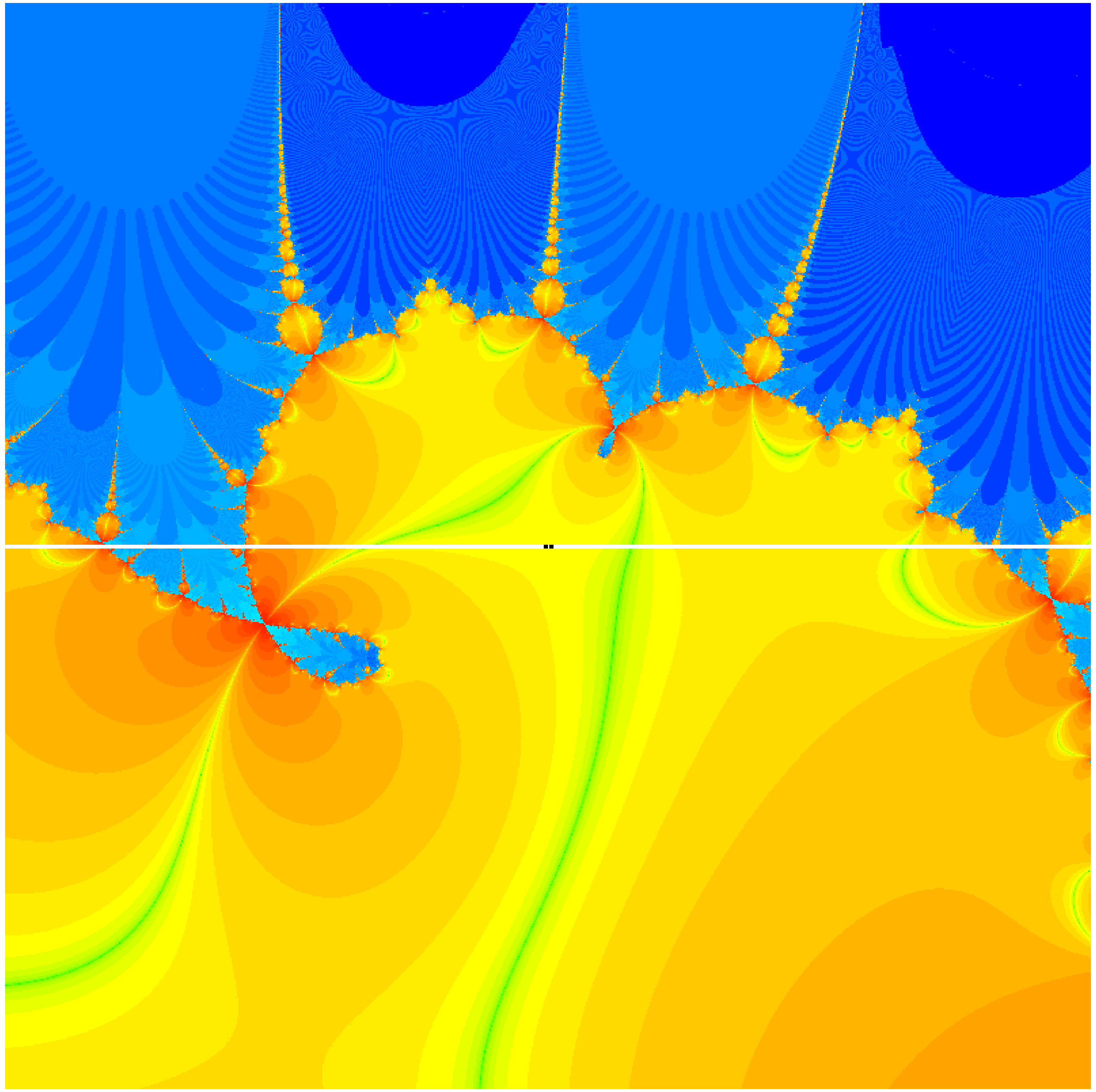
5. Clumps divide at points, s, where $\zeta(s)$ = RFP

6. Clump divisions are defined by contour lines $\mathrm{Im}(\zeta(s)) = 0$ that pass through Gram points

Lehmer pair: $0.5 + 663318.508i$ and $0.5 + 663318.511i \quad \delta = 0.0030$

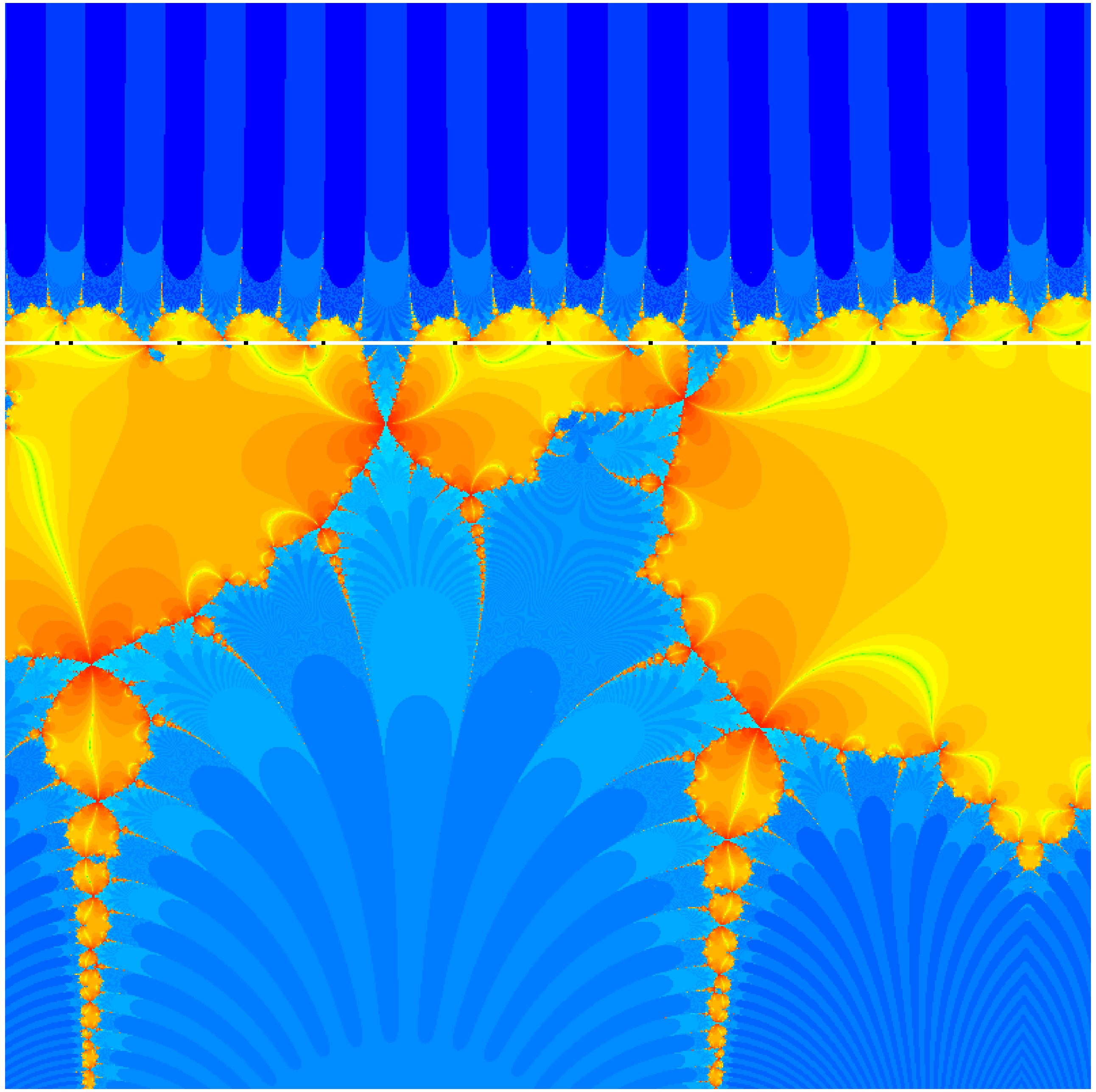


$-2 \leq \Re(s) \leq 6, 663314.5098 \leq \Im(s) \leq 663322.5098, 100ppu$

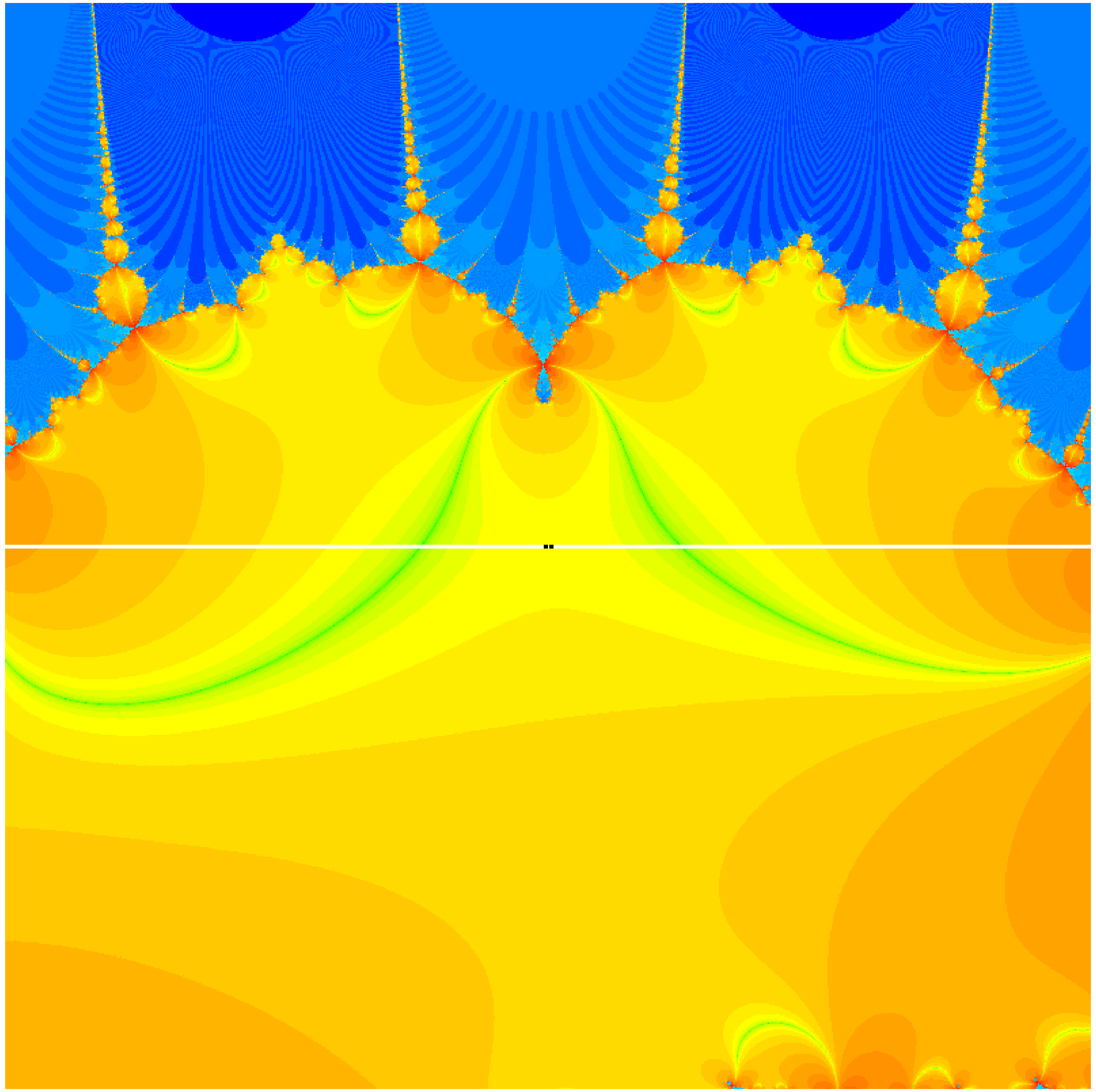


$0 \leq \Re(s) \leq 1, 663318.0098 \leq \Im(s) \leq 663318.0098, 800ppu$

Lehmer pair: $0.5 + 273193.663i$ and $0.5 + 273193.669i \quad \delta = 0.0057$

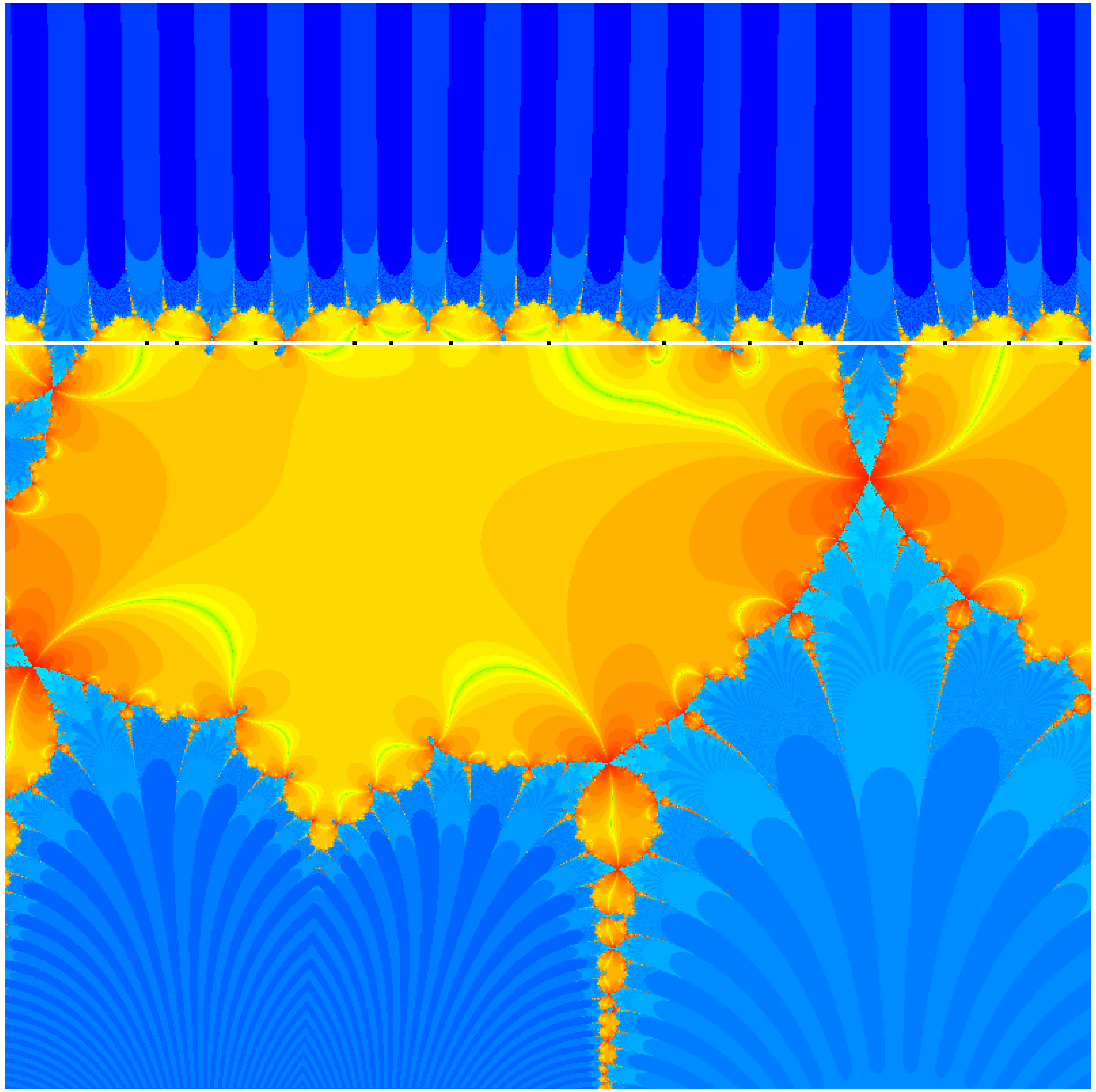


$-2 \le \Re(s) \le 6, 273189.666 \le \Im(s) \le 273197.666, 100ppu$

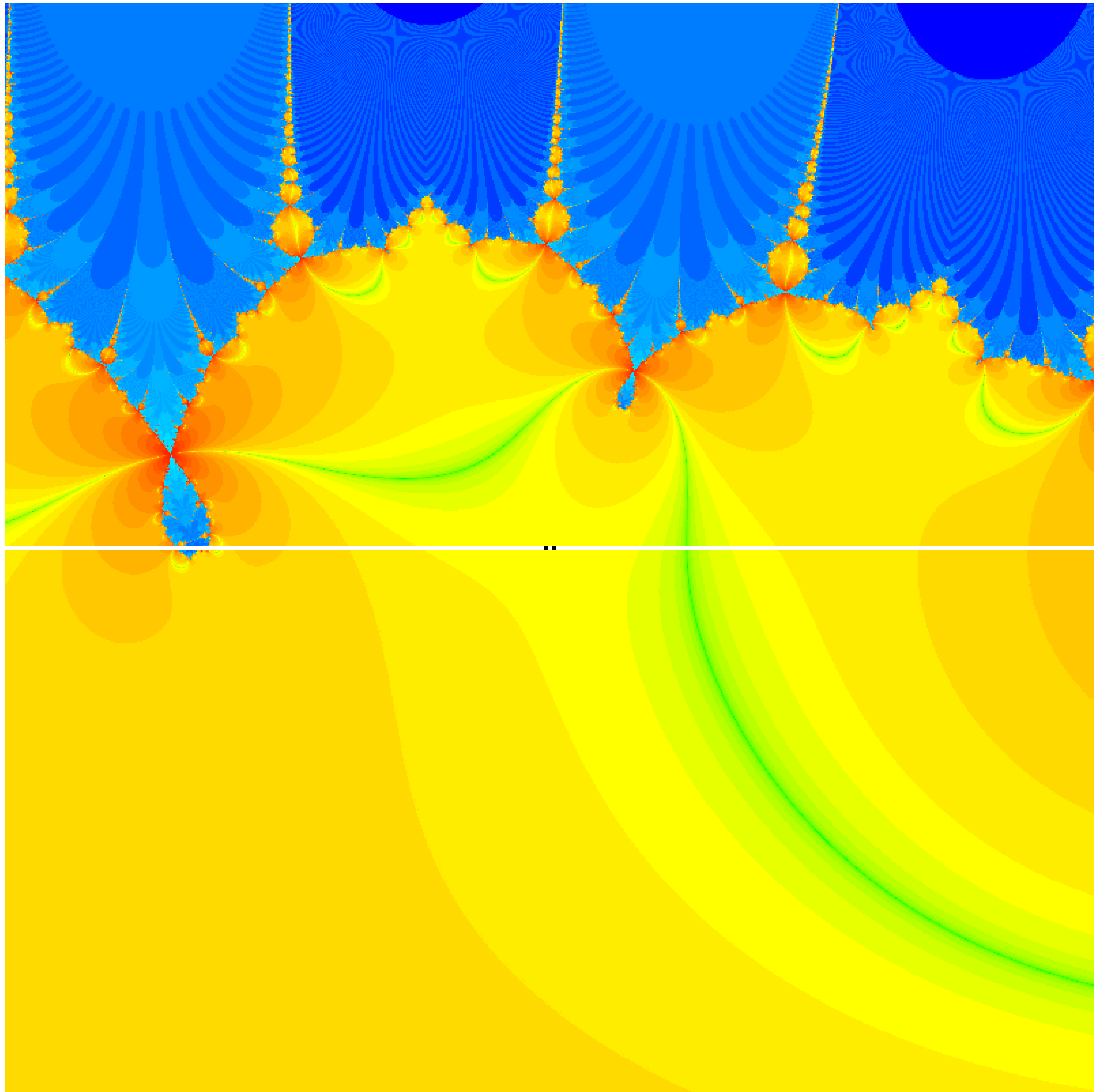


$0 \le \Re(s) \le 1, 273193.166 \le \Im(s) \le 273194.166, 800ppu$

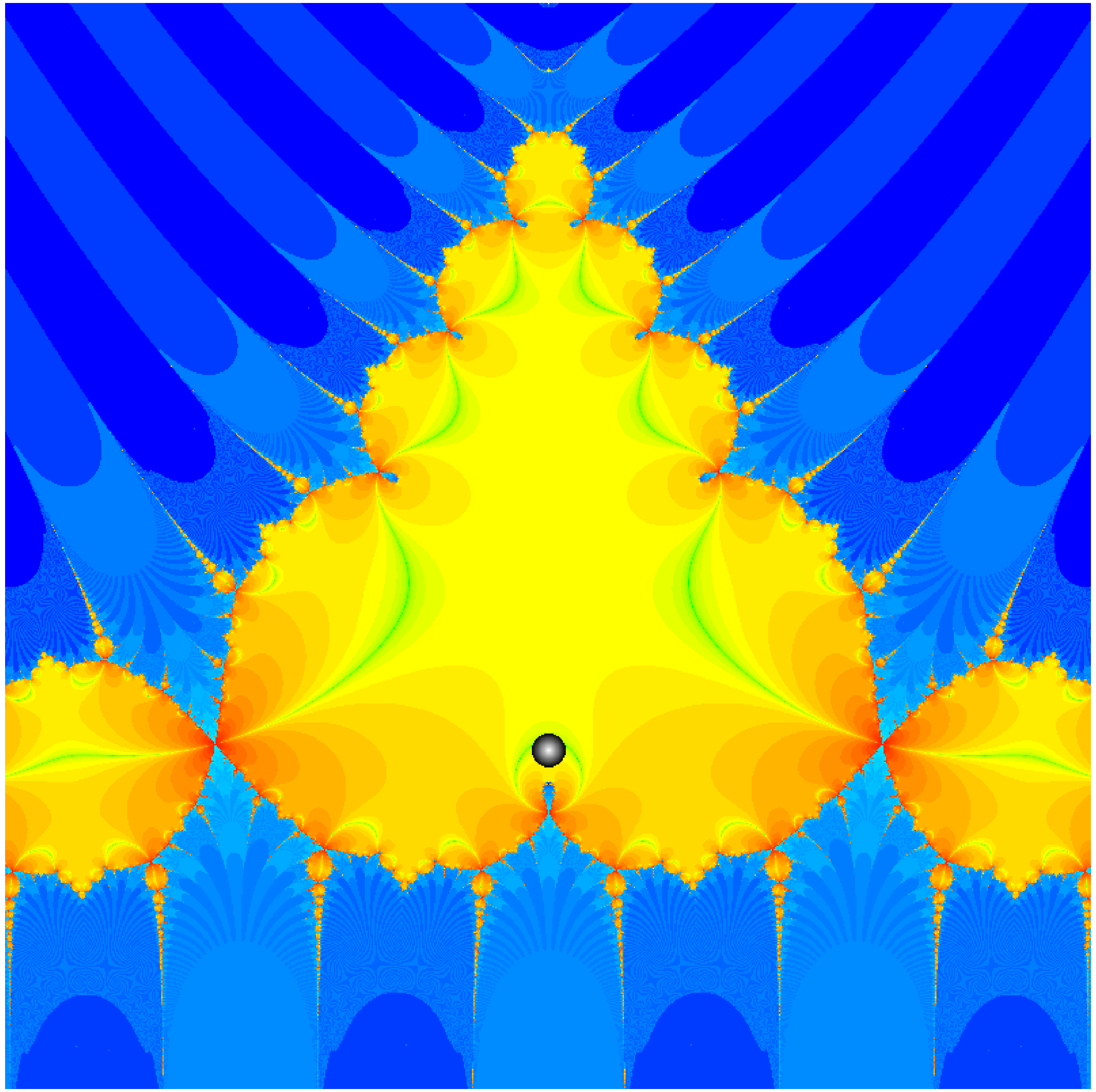
Lehmer pair: $0.5 + 712004.001i$ and $0.5 + 712004.007i$ $\quad \delta = 0.0063$

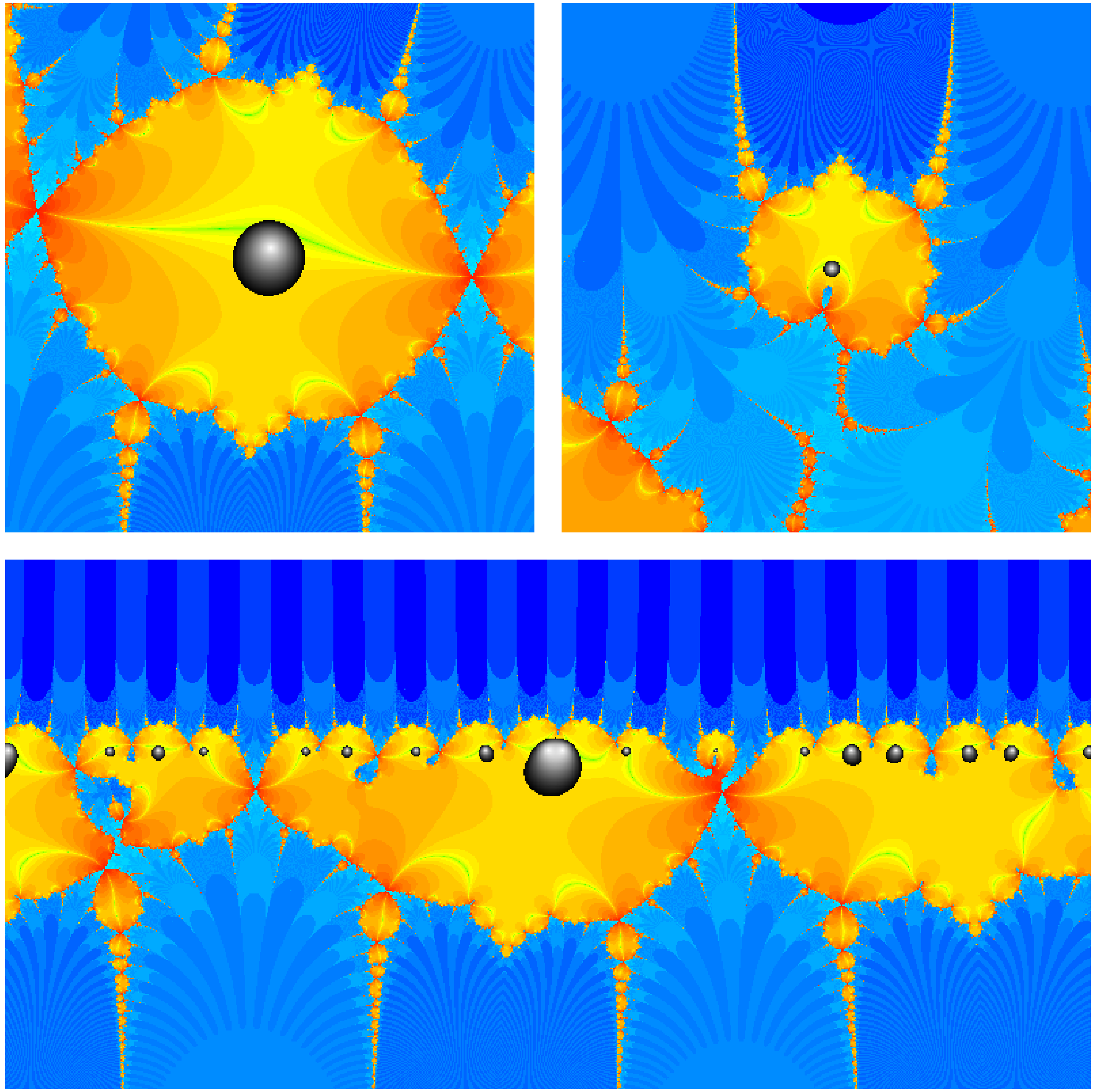


$-2 \leq \Re(s) \leq 6, 712000.0037 \leq \Im(s) \leq 712008.0037, 100ppu$

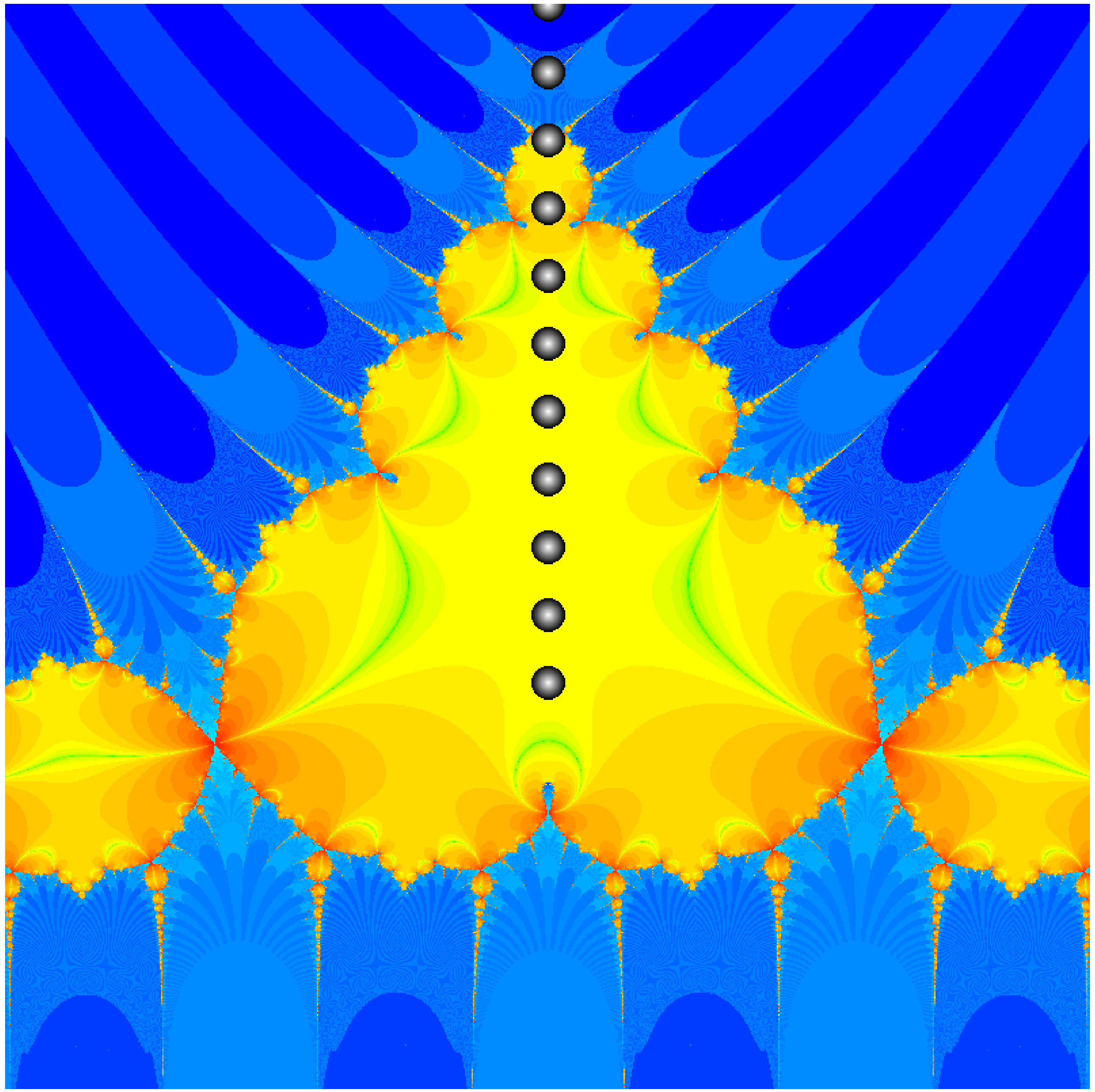


$0 \leq \Re(s) \leq 1, 712003.5037 \leq \Im(s) \leq 712004.5037, 800ppu$


$|s| < 0.5$
$-22 \le \Re(s) \le 10, -16 \le \Im(s) \le 16, 25ppu$


$|\zeta(s)| < 0.5$
$-4 \le \Re(s) \le 5.75, 9.25 \le \Im(s) \le 19, 40ppu$
$0 \le \Re(s) \le 1, 28978.9 \le \Im(s) \le 28979.9, 390ppu$
$-3 \le \Re(s) \le 6.75, 1600 \le \Im(s) \le 1620, 40ppu$

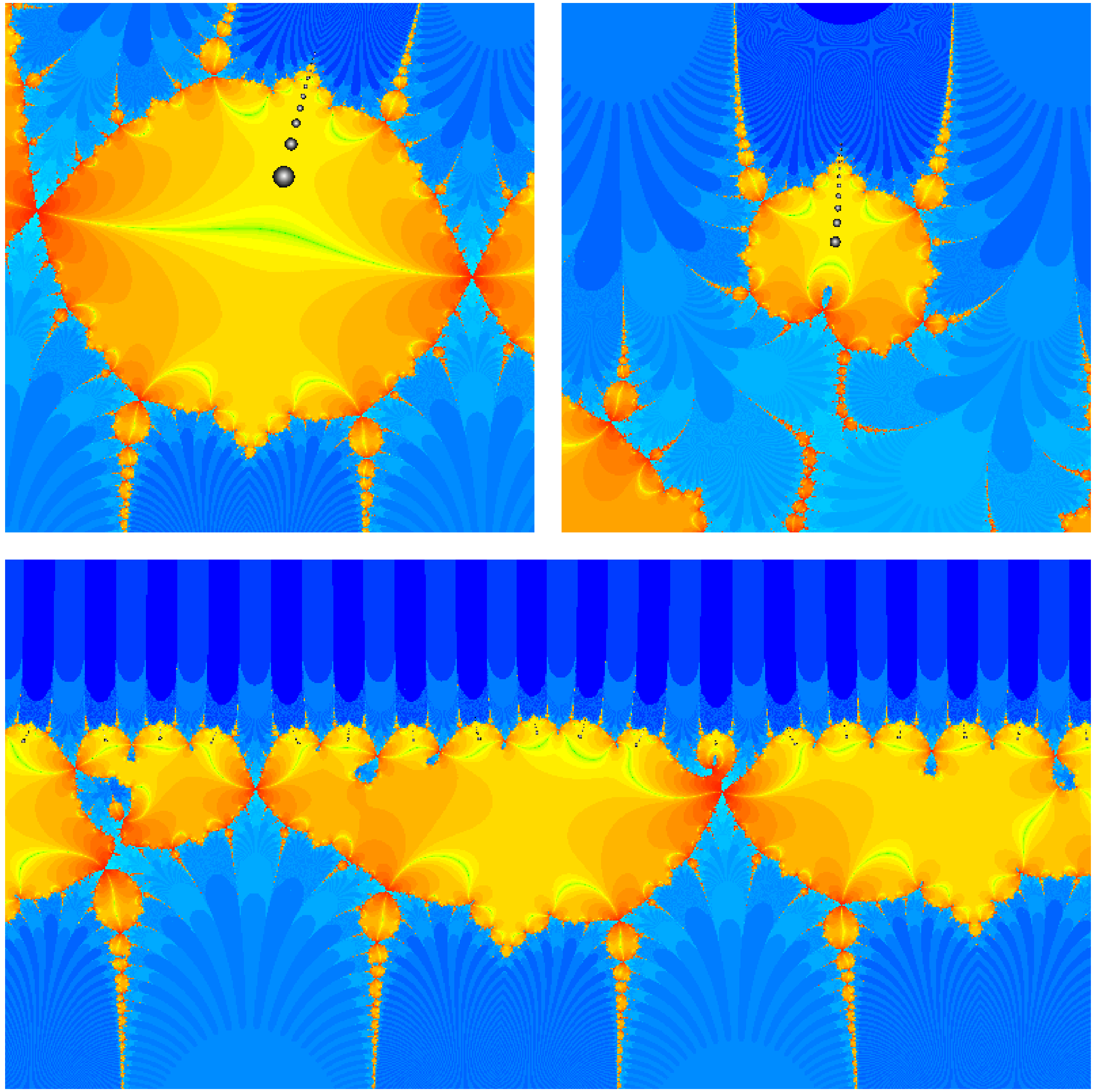
$$|s+2,+4,+6,...| < 0.5$$



$-22 \le \Re(s) \le 10, -16 \le \Im(s) \le 16, 25ppu$

$$|\zeta(s)+2,+4,+6,...| < 0.5$$

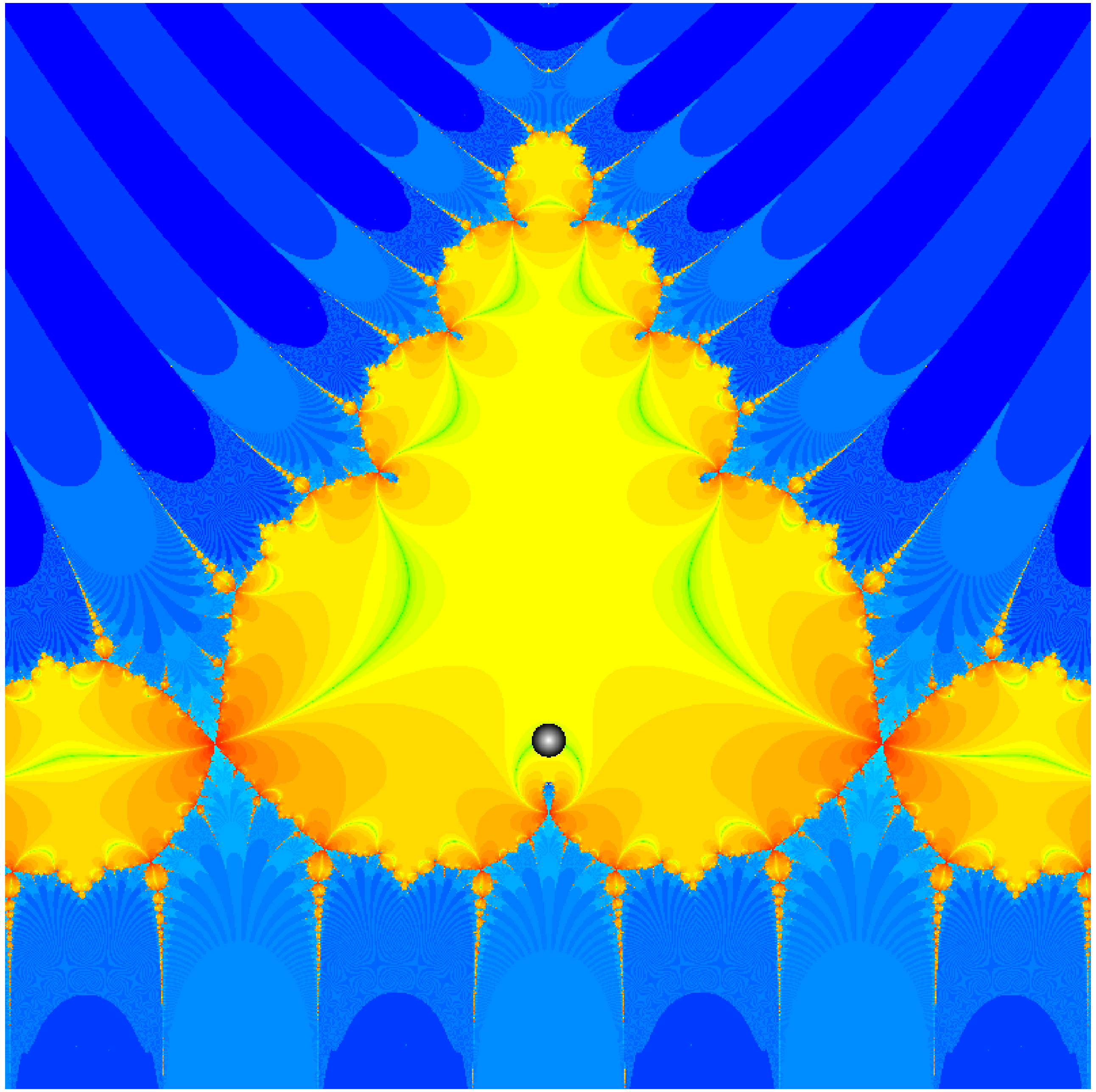
$-4 \le \Re(s) \le 5.75, 9.25 \le \Im(s) \le 19, 40ppu$



$0 \le \Re(s) \le 1, 28978.9 \le \Im(s) \le 28979.9, 390ppu$

$-3 \le \Re(s) \le 6.75, 1600 \le \Im(s) \le 1620, 40ppu$

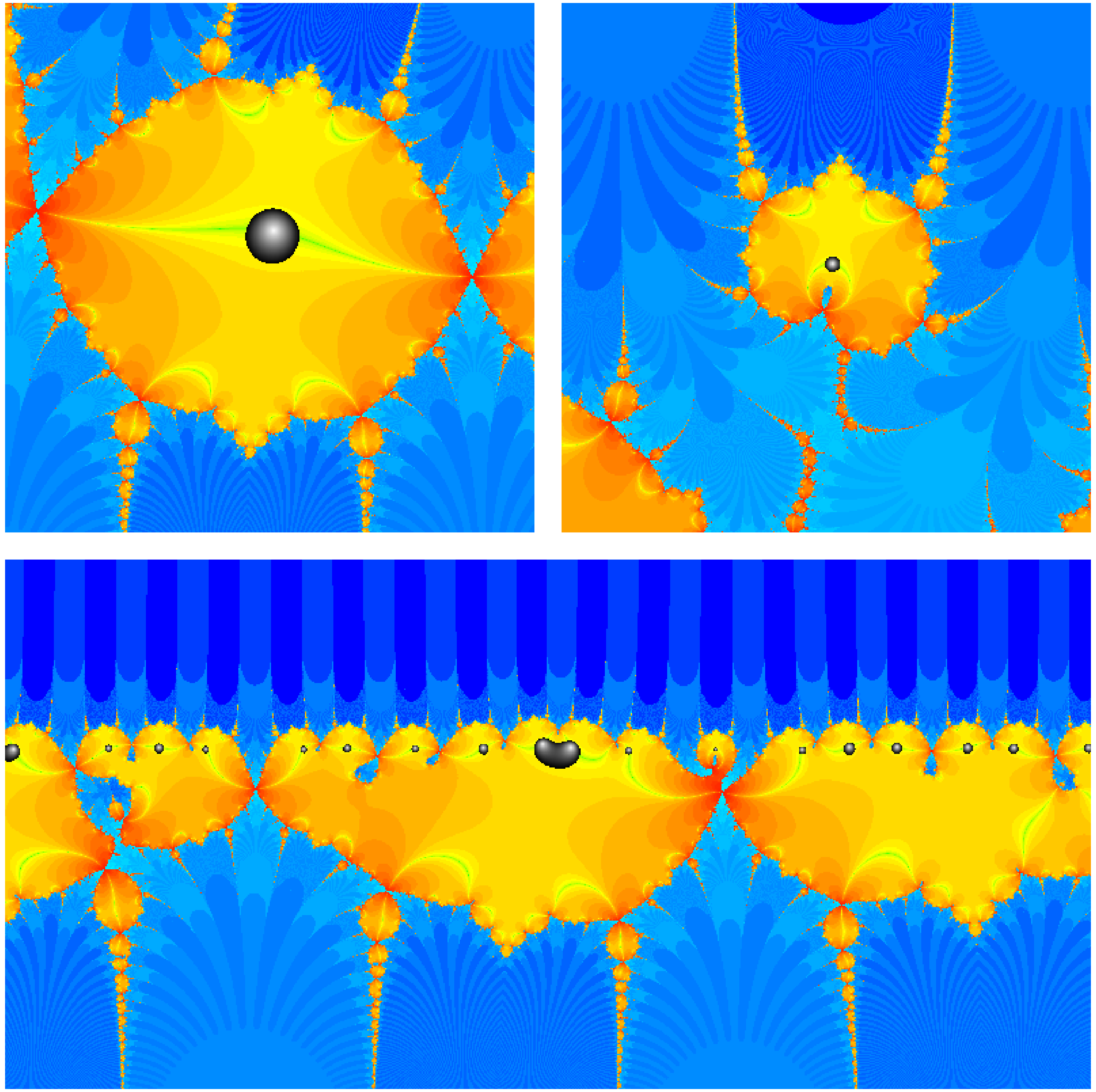
$$|s + 0.295905| < 0.5$$



$-22 \leq \Re(s) \leq 10, -16 \leq \Im(s) \leq 16, 25ppu$

$$|\zeta(s) + 0.295905| < 0.5$$

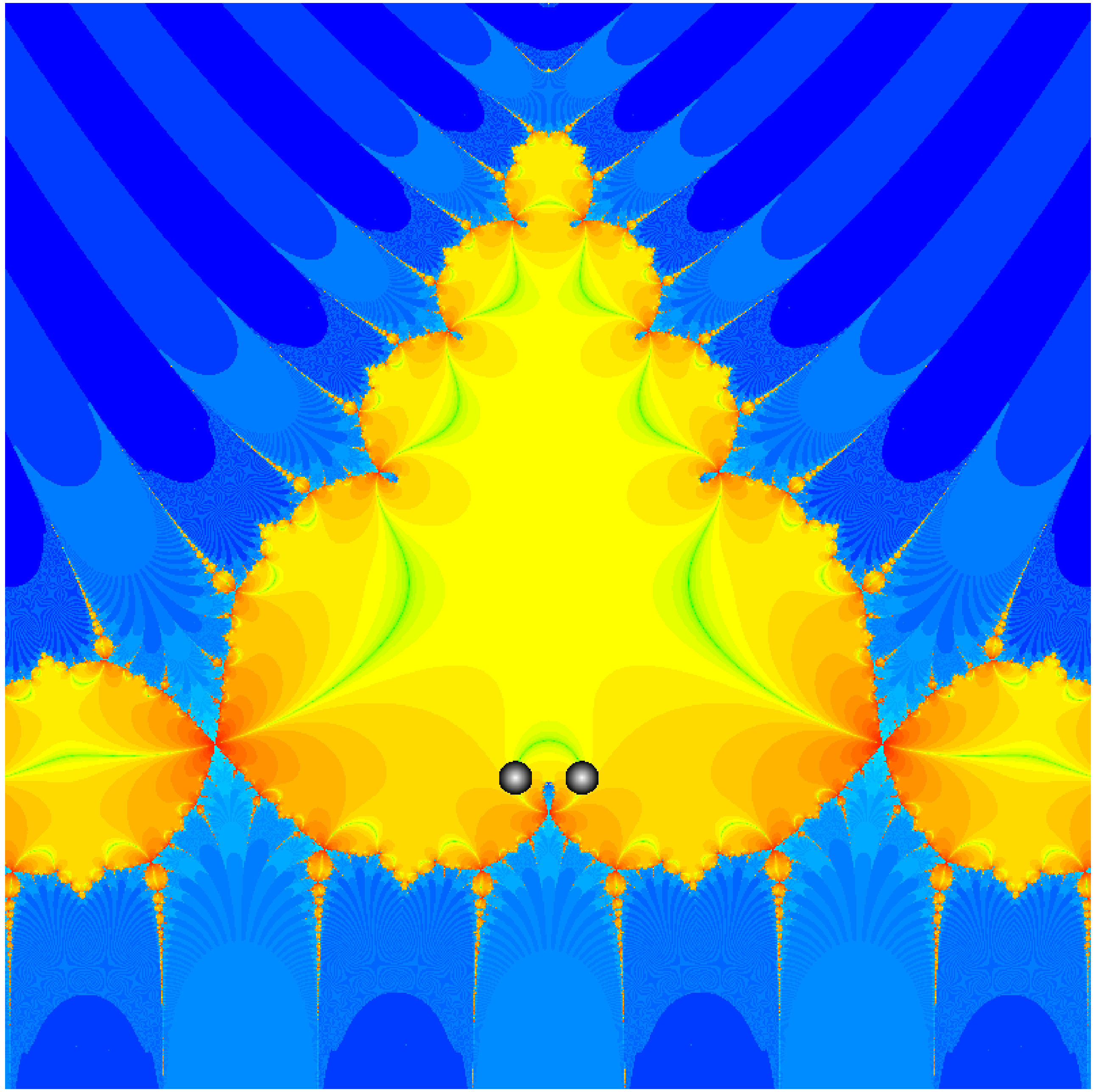
$-4 \leq \Re(s) \leq 5.75, 9.25 \leq \Im(s) \leq 19, 40ppu$



$0 \leq \Re(s) \leq 1, 28978.9 \leq \Im(s) \leq 28979.9, 390ppu$

$-3 \leq \Re(s) \leq 6.75, 1600 \leq \Im(s) \leq 1620, 40ppu$

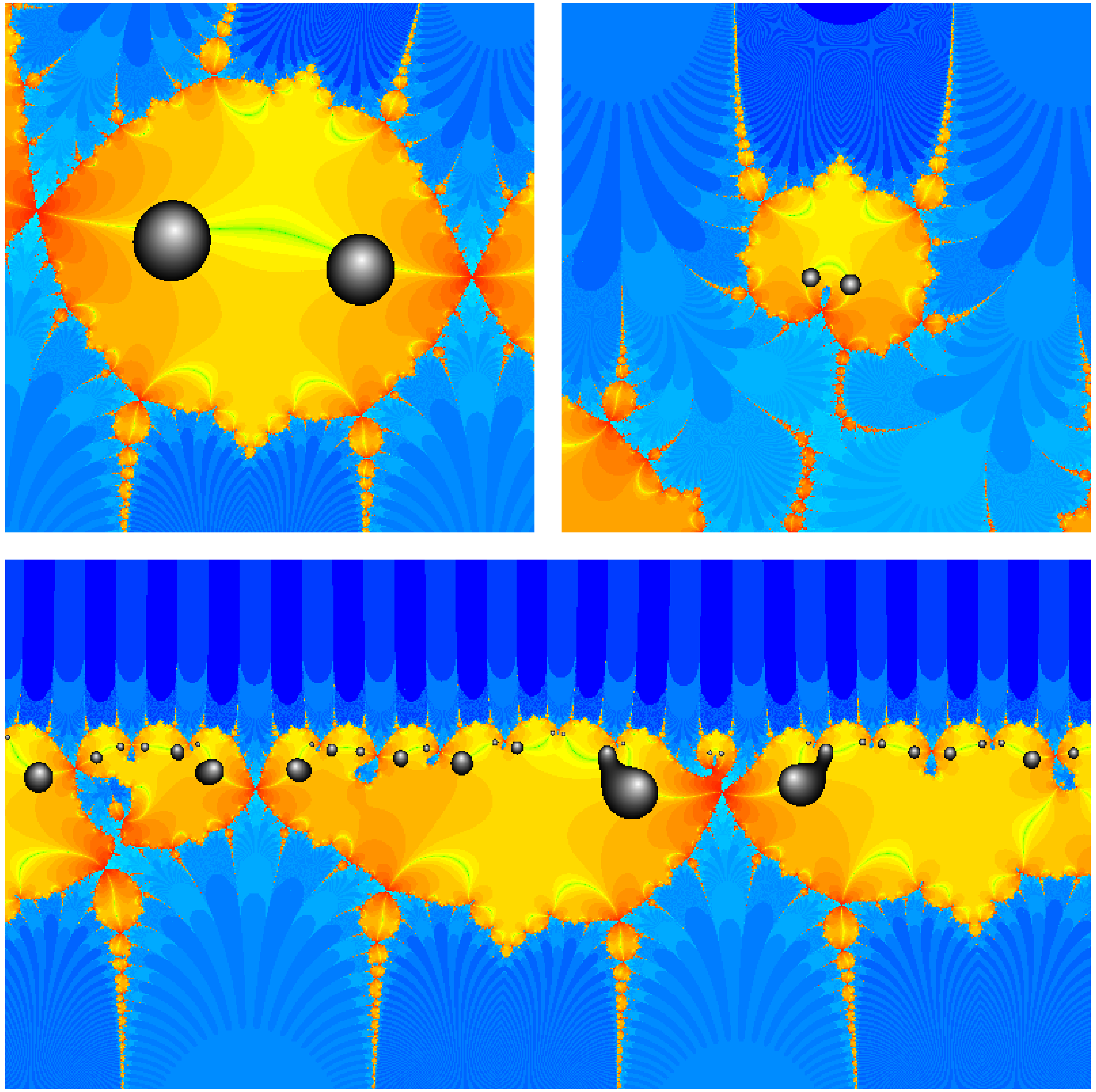
$$|s - (0.81 \pm 0.98i)| < 0.5$$



$-22 \leq \Re(s) \leq 10, -16 \leq \Im(s) \leq 16, 25ppu$

$$|\zeta(s) - (0.81 \pm 0.98i)| < 0.5$$

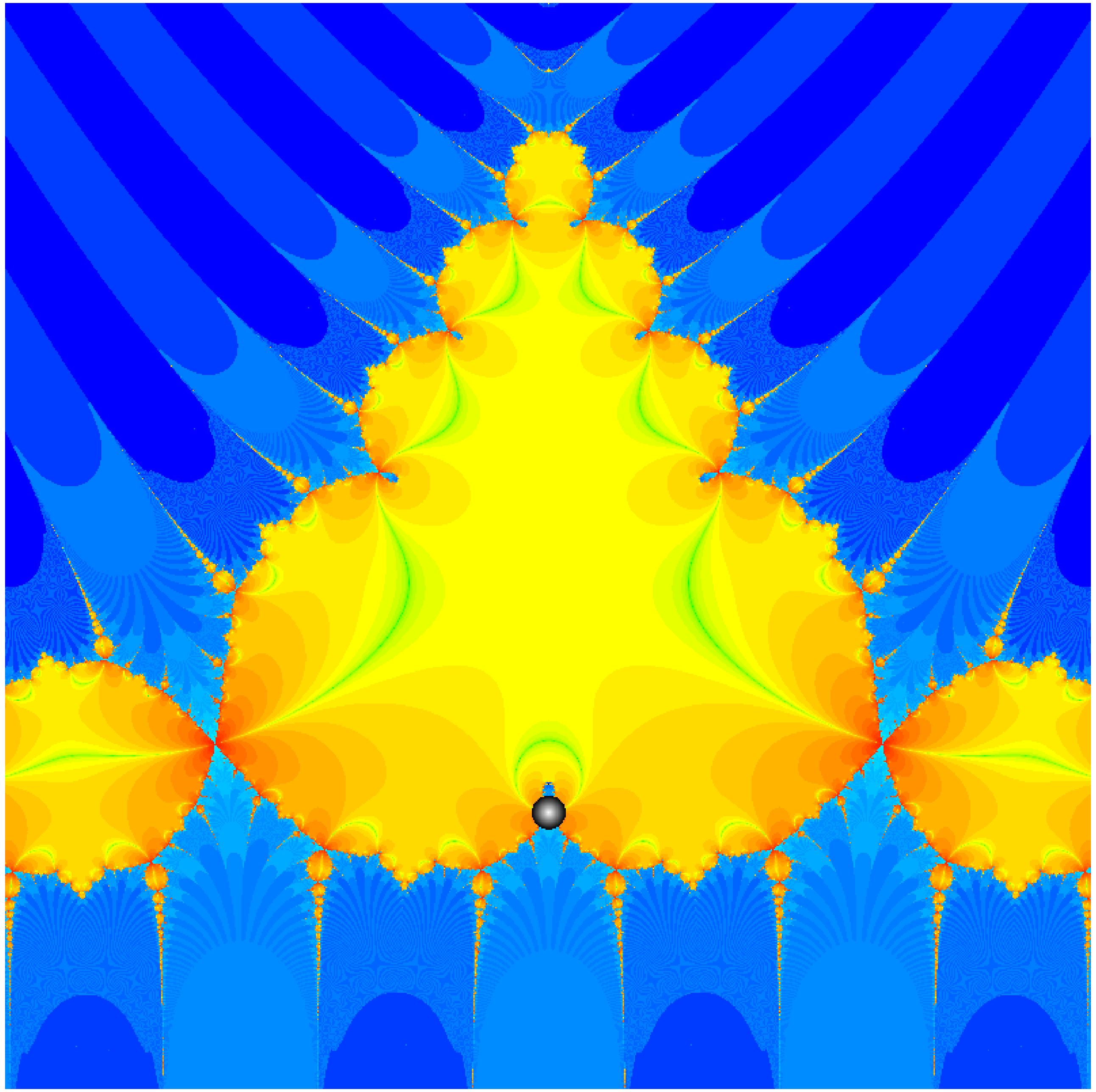
$-4 \leq \Re(s) \leq 5.75, 9.25 \leq \Im(s) \leq 19, 40ppu$



$0 \leq \Re(s) \leq 1, 28978.9 \leq \Im(s) \leq 28979.9, 390ppu$

$-3 \leq \Re(s) \leq 6.75, 1600 \leq \Im(s) \leq 1620, 40ppu$

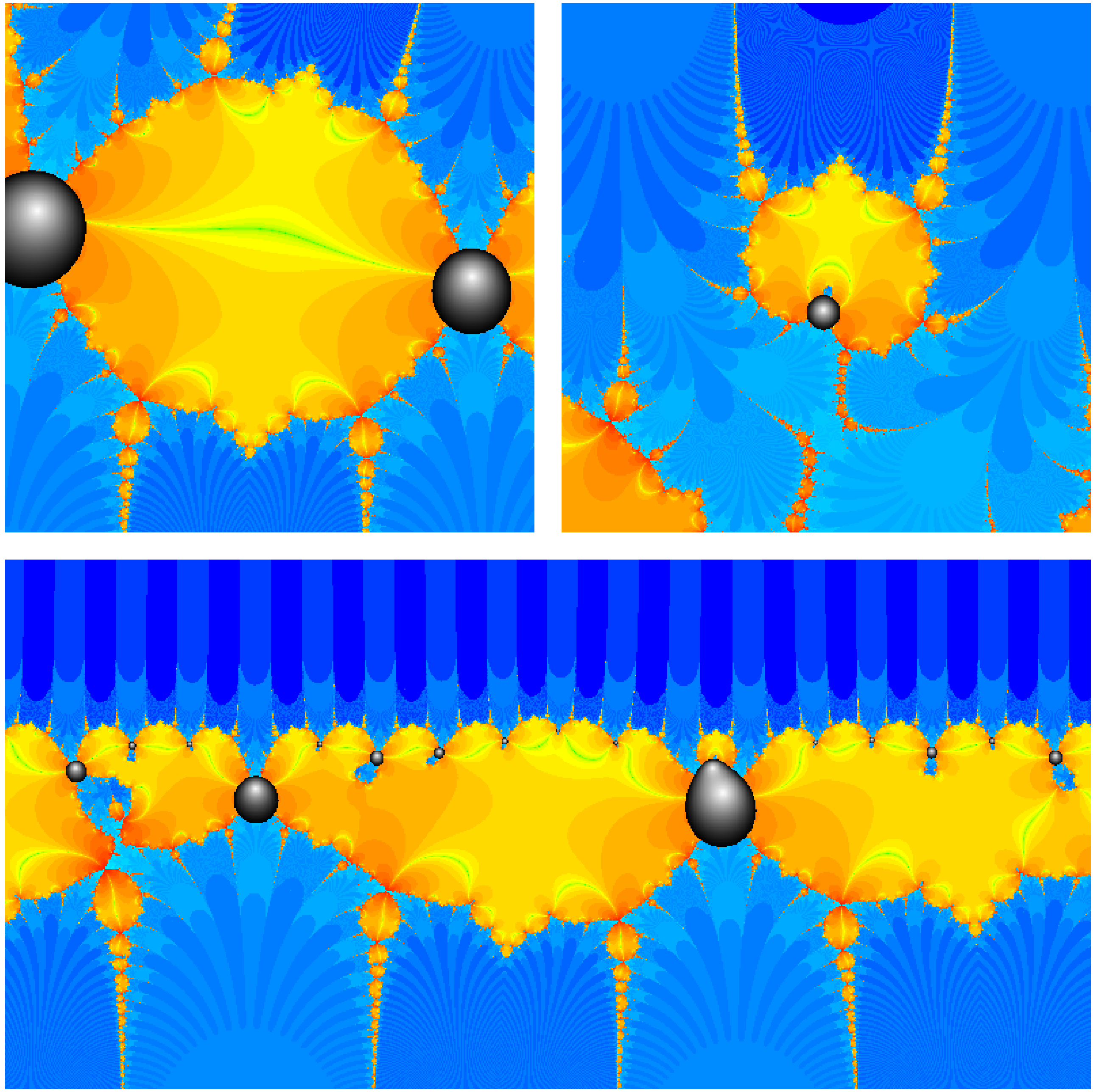
$|s - 1.83377| < 0.5$



$-22 \leq \Re(s) \leq 10, -16 \leq \Im(s) \leq 16, 25ppu$

$|\zeta(s) - 1.83377| < 0.5$

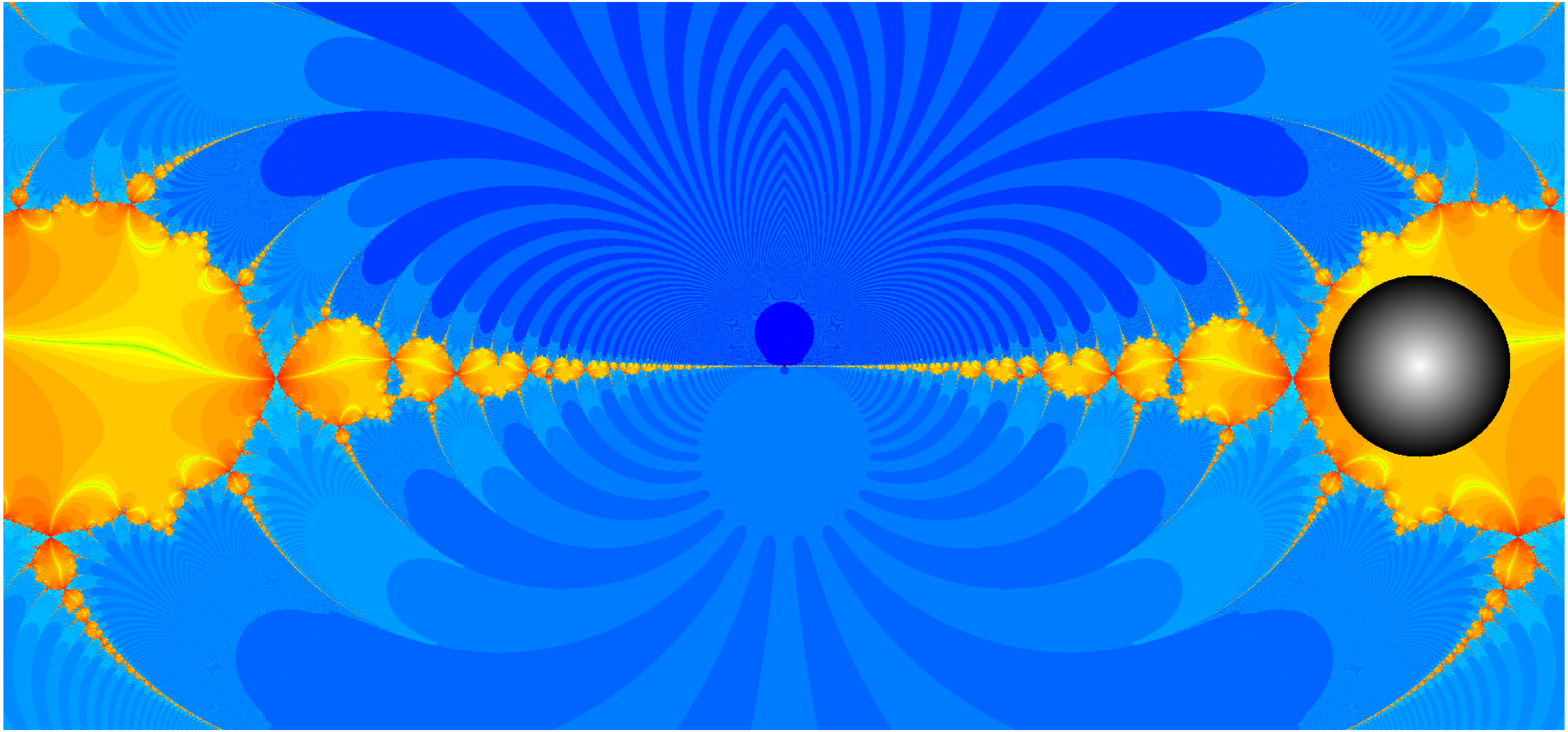
$-4 \leq \Re(s) \leq 5.75, 9.25 \leq \Im(s) \leq 19, 40ppu$



$0 \leq \Re(s) \leq 1, 28978.9 \leq \Im(s) \leq 28979.9, 390ppu$

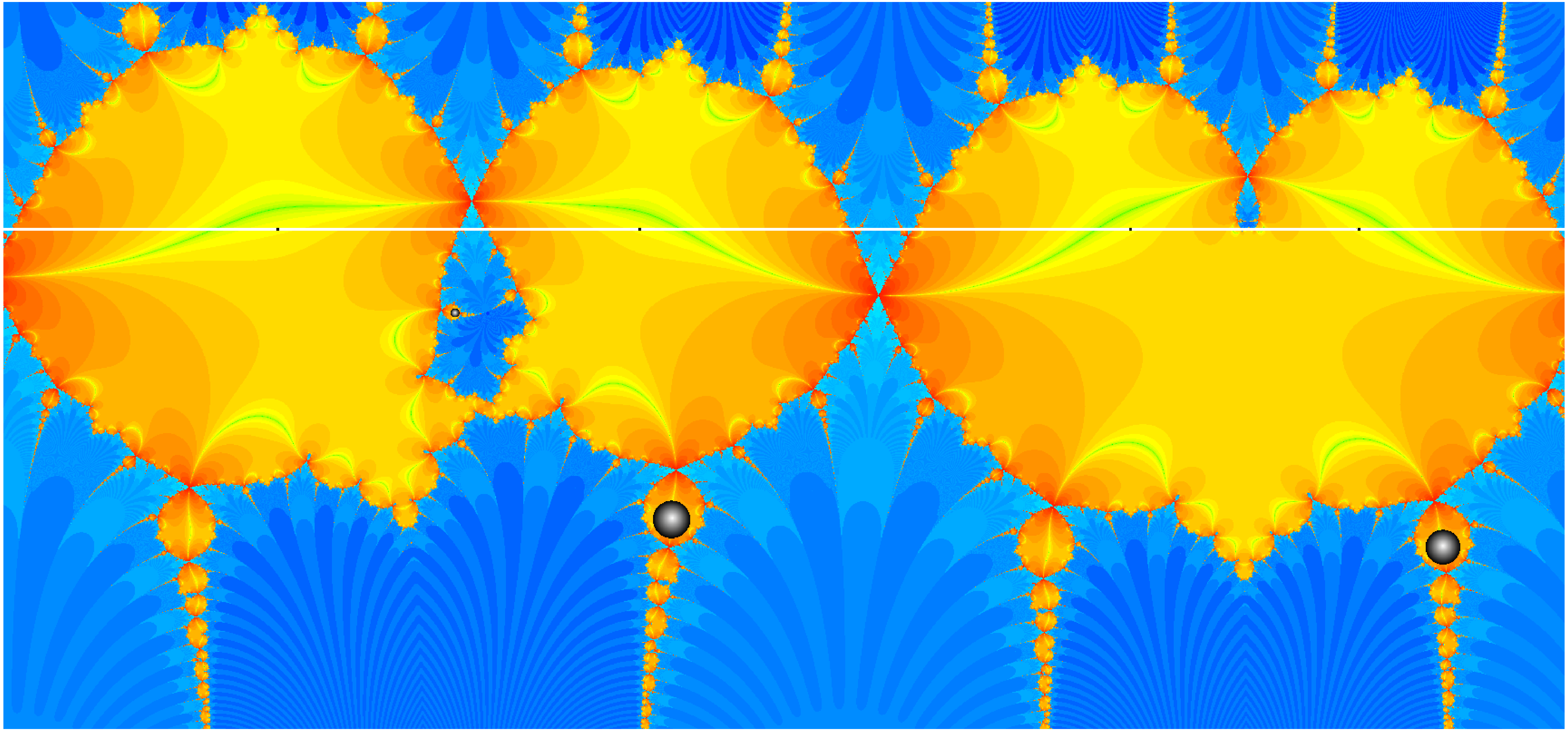
$-3 \leq \Re(s) \leq 6.75, 1600 \leq \Im(s) \leq 1620, 40ppu$

$$|s-(1+0.07i)|<0.01$$



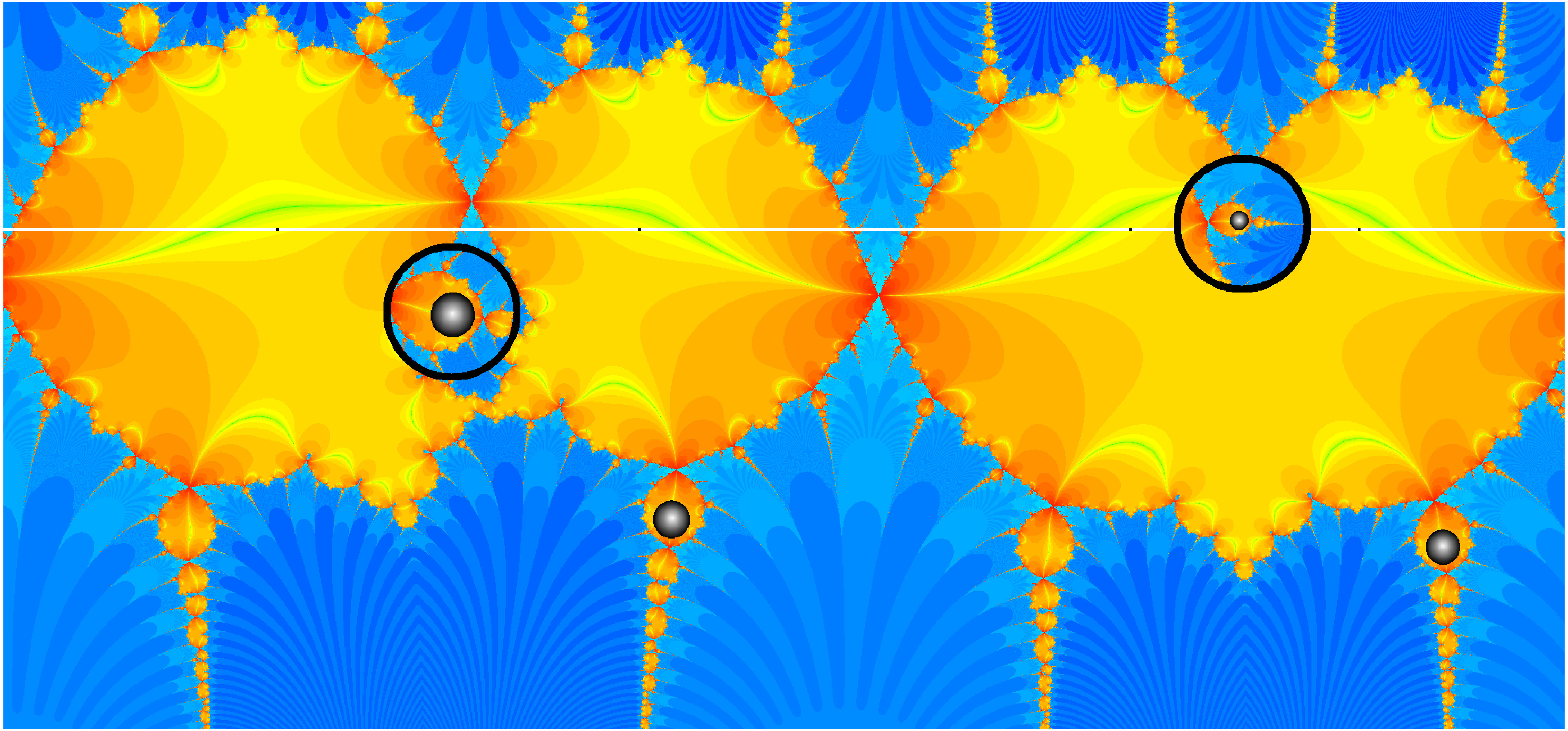
$0.96 \leq \Re(s) \leq 1.04, -0.086 \leq \Im(s) \leq 0.086, 10000ppu$

$$|\zeta(s) - (1 + 0.07i)| < 0.01$$



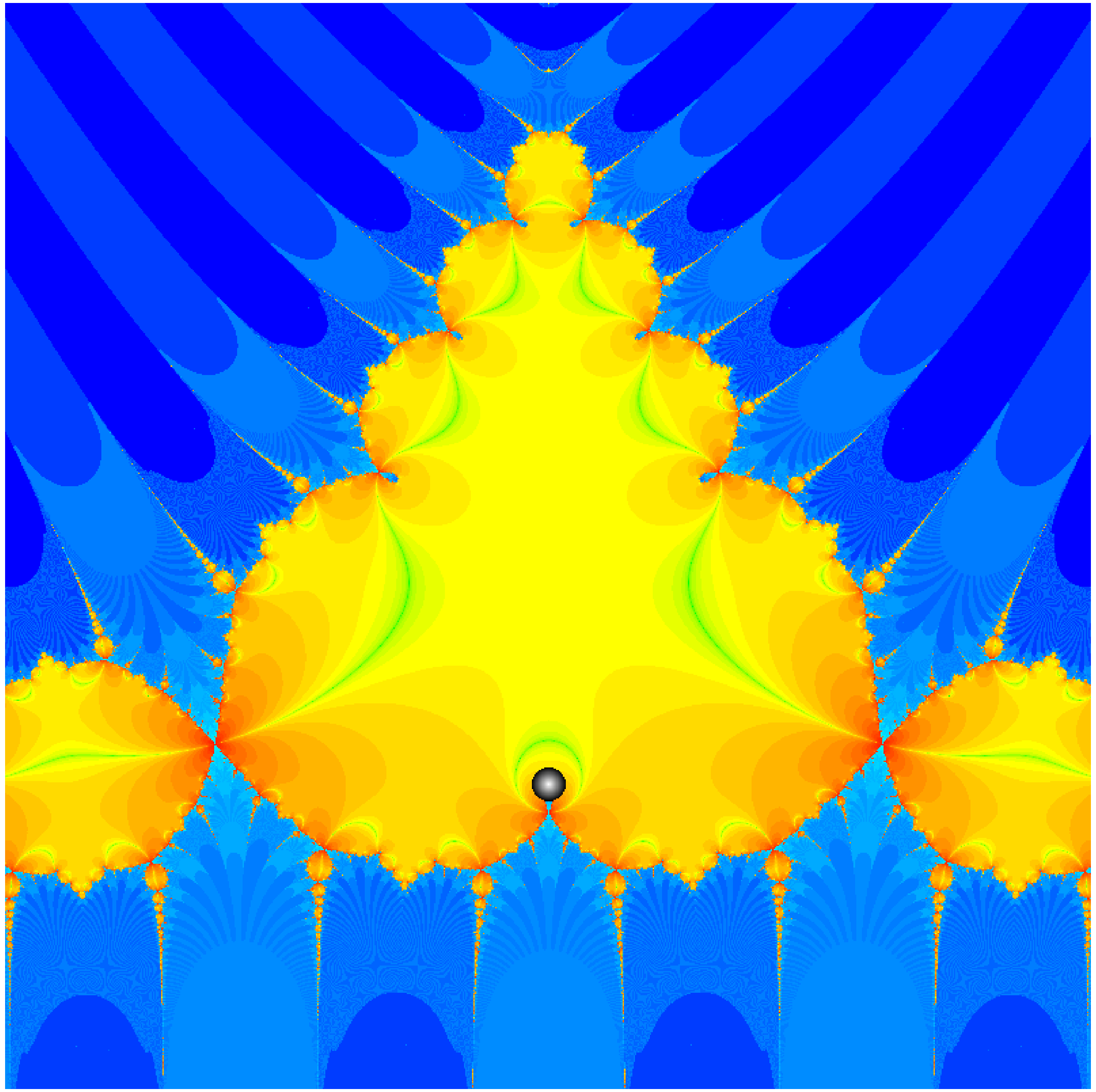
$-2 \le \Re(s) \le 6, 18 \le \Im(s) \le 35.2, 100ppu$

$$|\zeta(s) - (1 + 0.07i)| < 0.01$$



$-2 \leq \Re(s) \leq 6, 18 \leq \Im(s) \leq 35.2, 100ppu$

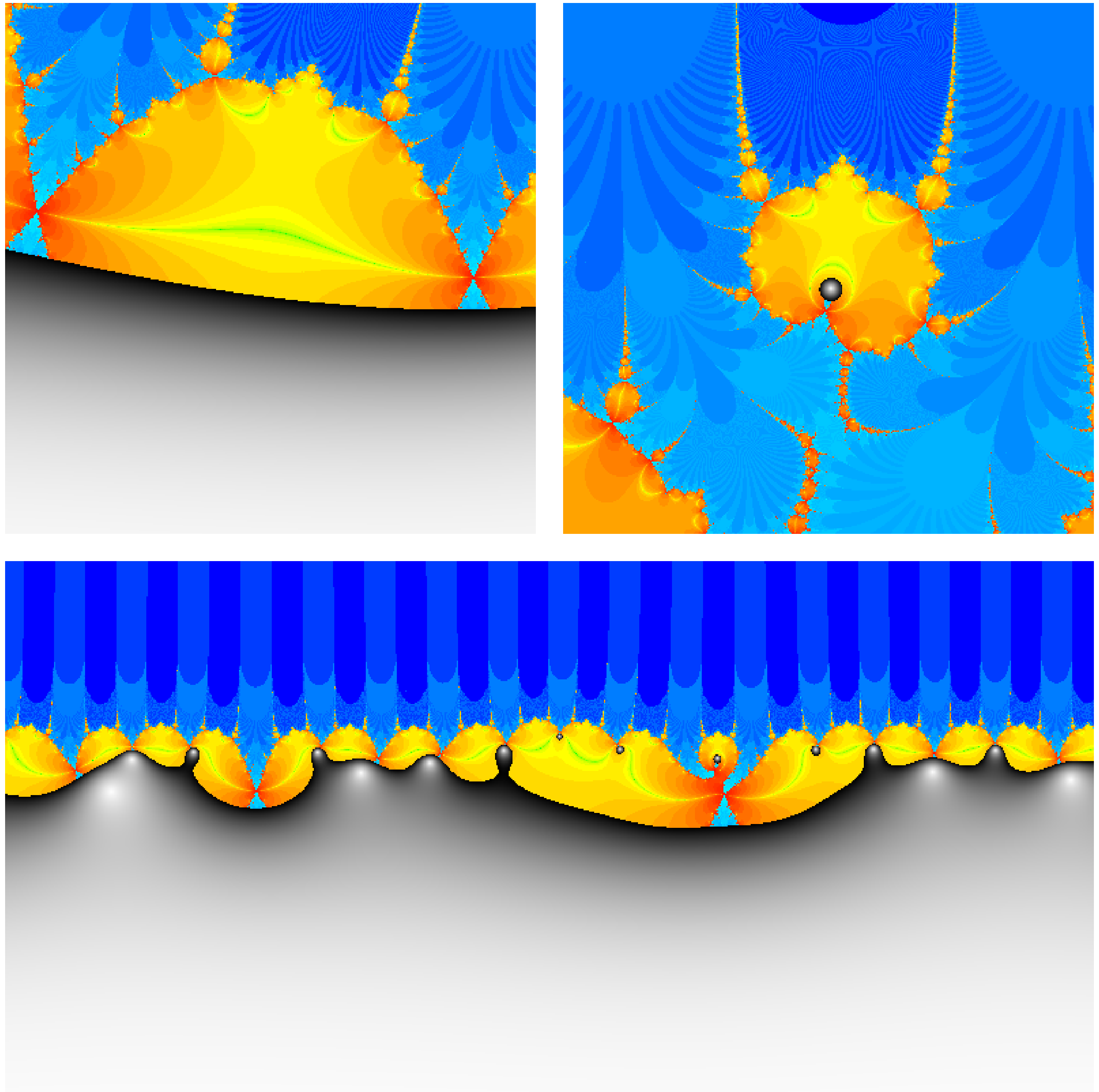
$|s-1| < 0.5$



$-22 \le \Re(s) \le 10, -16 \le \Im(s) \le 16, 25ppu$

$|\zeta(s)-1| < 0.5$

$-4 \le \Re(s) \le 5.75, 9.25 \le \Im(s) \le 19, 40ppu$



$0 \le \Re(s) \le 1, 28978.9 \le \Im(s) \le 28979.9, 390ppu$

$-1 \le \Re(s) \le 25, 0 \le \Im(s) \le 25, 32ppu$

X-Ray

X-RAY OF RIEMANN'S ZETA-FUNCTION

J. ARIAS-DE-REYNA

1. INTRODUCTION

This paper is the result of the effort to give the students of the subject *Analytic Number Theory* an idea of the complexity of the behaviour of the Riemann zeta-function. I tried to make them **see** with their own eyes the mystery contained in its apparently simple definition.

There are precedents for the figures we are about to present. In the tables of Jahnke-Emde [9] we can find pictures of the zeta-function and some other graphs where we can see some of the lines we draw. In the dissertation of A. Utzinger [21], directed by Speiser, the lines $\operatorname{Re}\zeta(s) = 0$ and $\operatorname{Im}\zeta(s) = 0$ are drawn on the rectangle $(-9, 10) \times (0, 29)$.

Besides, Speiser's paper contains some very interesting ideas. He proves that the Riemann Hypothesis is equivalent to the fact that the non trivial zeros of $\zeta'(s)$ are on the right of the critical line. He proves this claim using an entirely geometric reasoning that is on the borderline between the proved and the admissible. Afterwards rigorous proofs of this statement have been given.

Our figures arise from a simple idea. If $f(z) = u(z) + iv(z)$ is a meromorphic function, then the curves $u = 0$ and $v = 0$ meet precisely at the zeros and poles of the function. That is the reason why we mark the curves where the function is real or the curves where it is imaginary on the z-plane. In order to distinguish one from the other, we will draw with thick lines the curves where the function is real and with thin lines the curves where the function is imaginary.

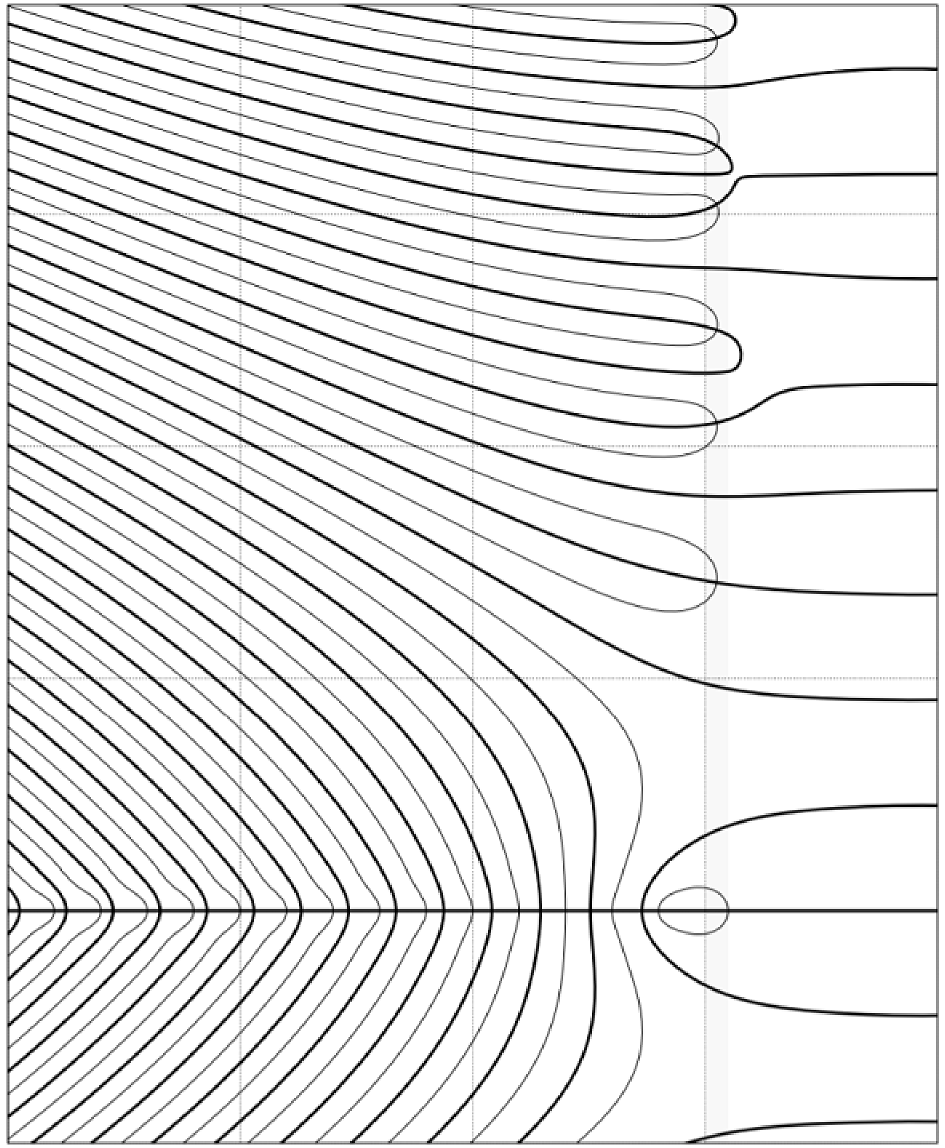
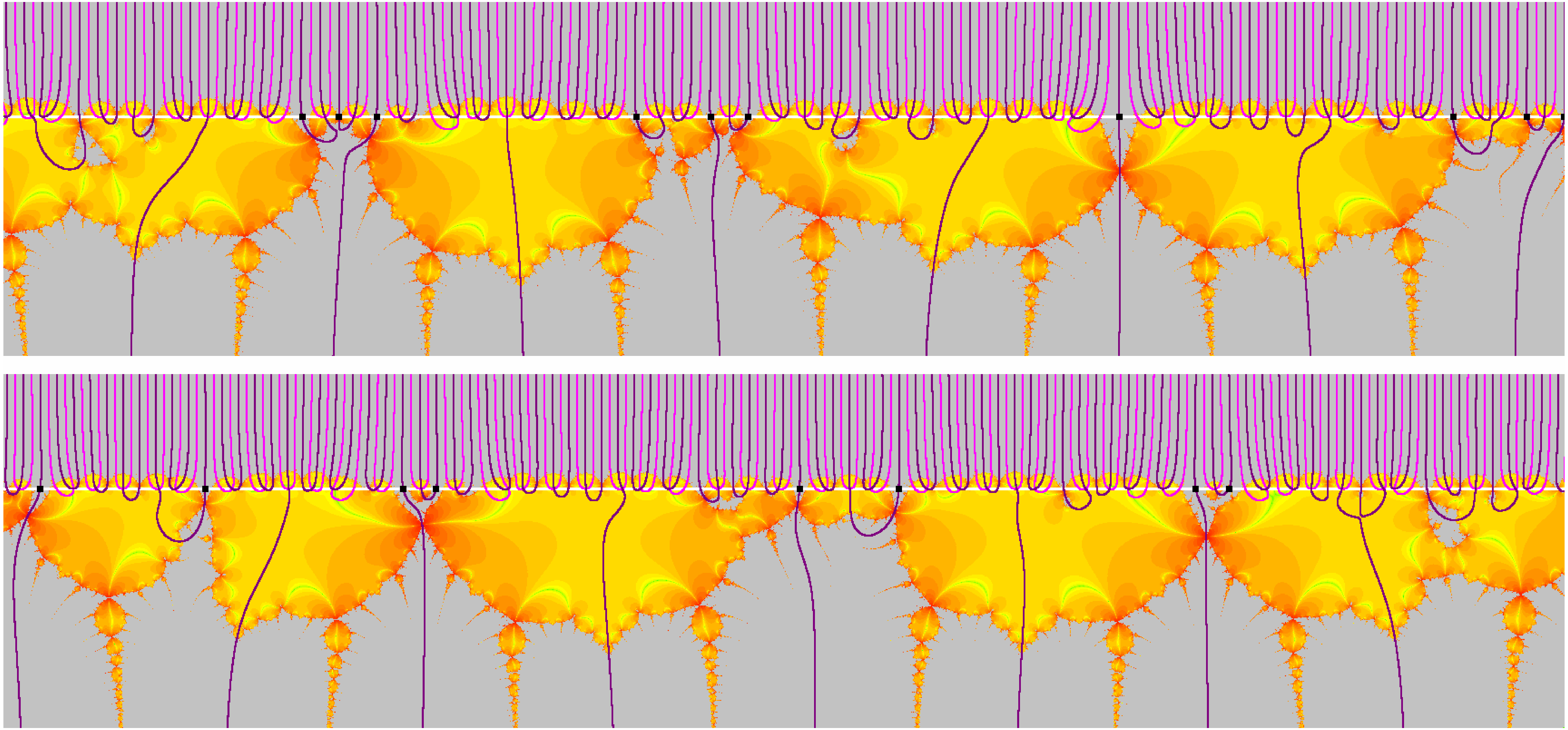


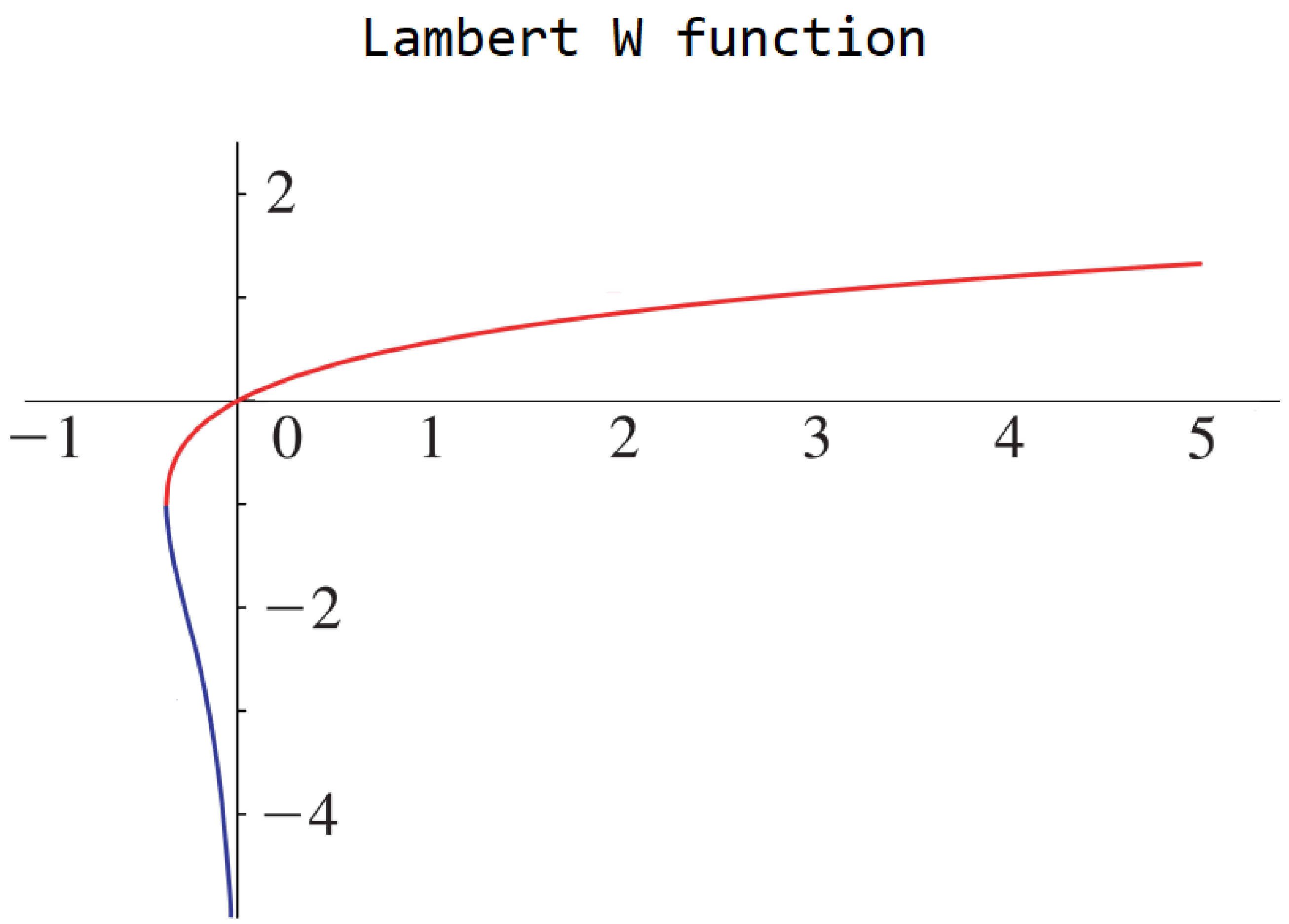
FIGURE 1. X ray of $\zeta(s)$

X-Ray

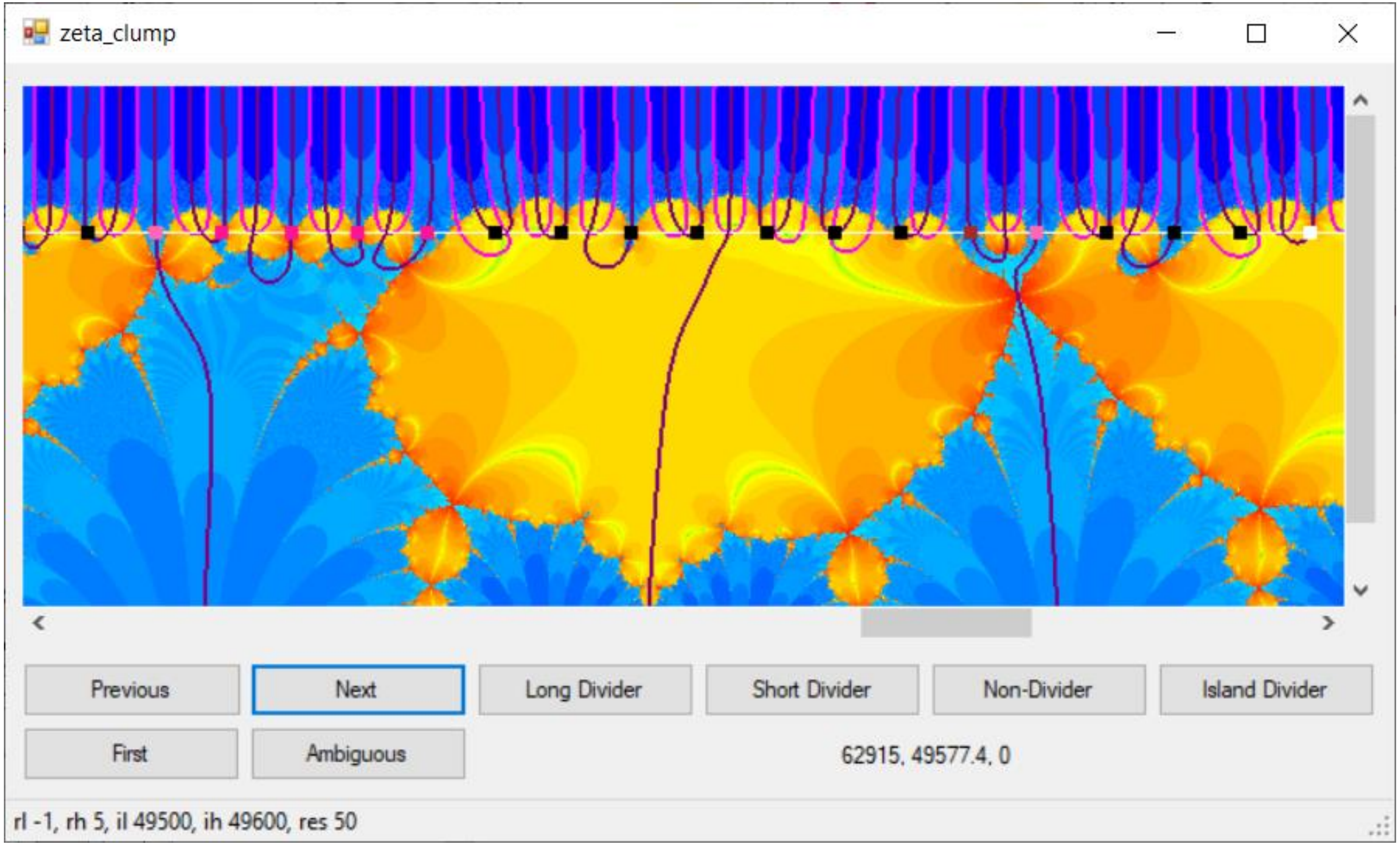


Gram point approximation via the Lambert W function

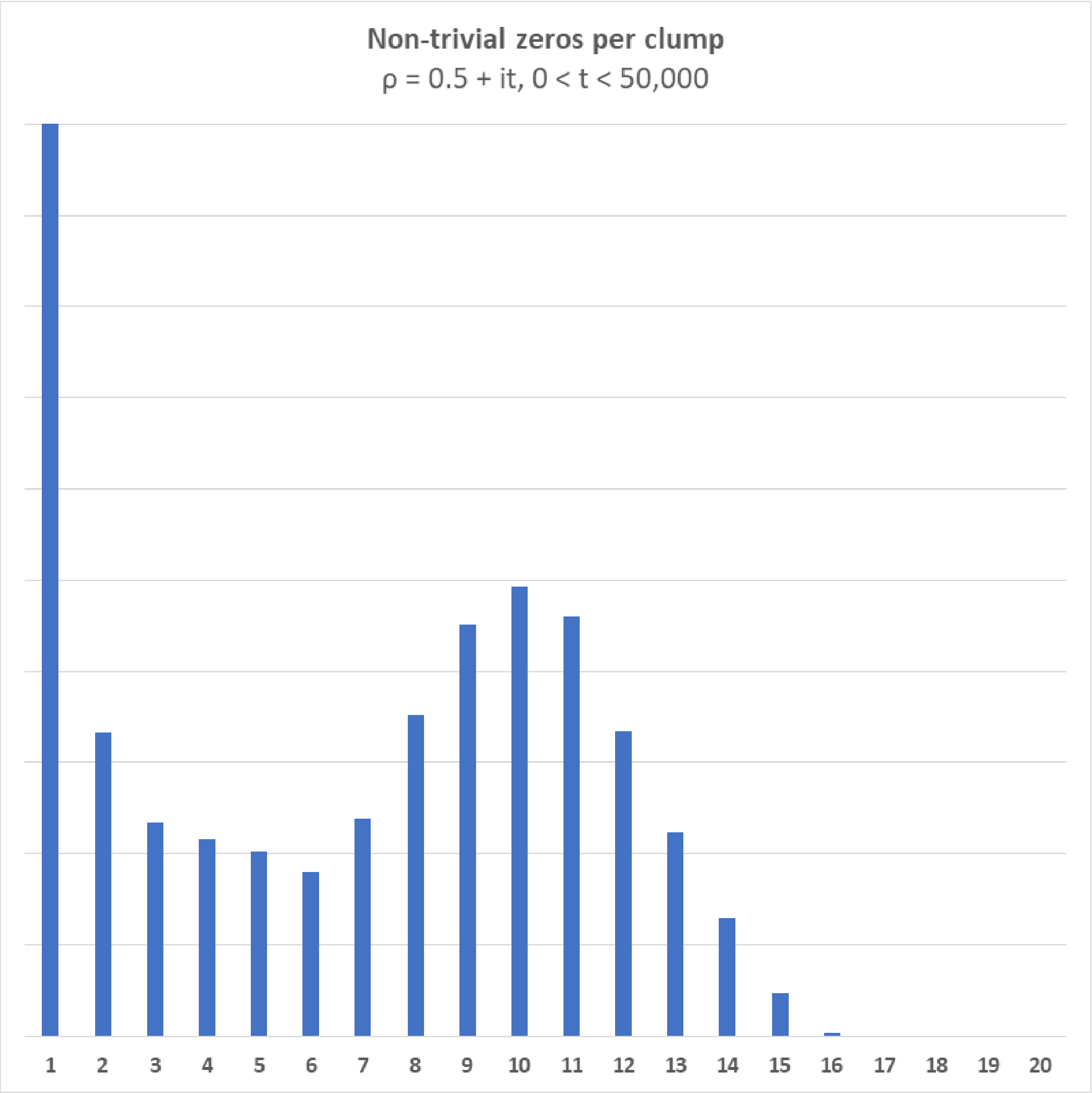
$$g_n \approx 2\pi \exp\left[1 + W\left(\frac{8n+1}{8e}\right)\right]$$



Clumps


zeta_clump
Previous
Next
Long Divider
Short Divider
Non-Divider
Island Divider
First
Ambiguous
62915, 49577.4, 0
rl -1, rh 5, il 49500, ih 49600, res 50

Clumps



Non-trivial zeros per clump
ρ = 0.5 + it, 0 < t < 50,000
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Hurwitz zeta functions

The Hurwitz zeta function $\zeta(s,a)$, for $s,a \in \mathbb{C}$ is defined as:

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(a+n)^s} \qquad \Re(s) > 1, \Re(a) > 0 \tag{1}$$

For $s \in \mathbb{C}, a \in \mathbb{Q}$, Hurwitz zeta functions have an analytic continuation to the whole complex plane other than a simple pole at $s = 1$:

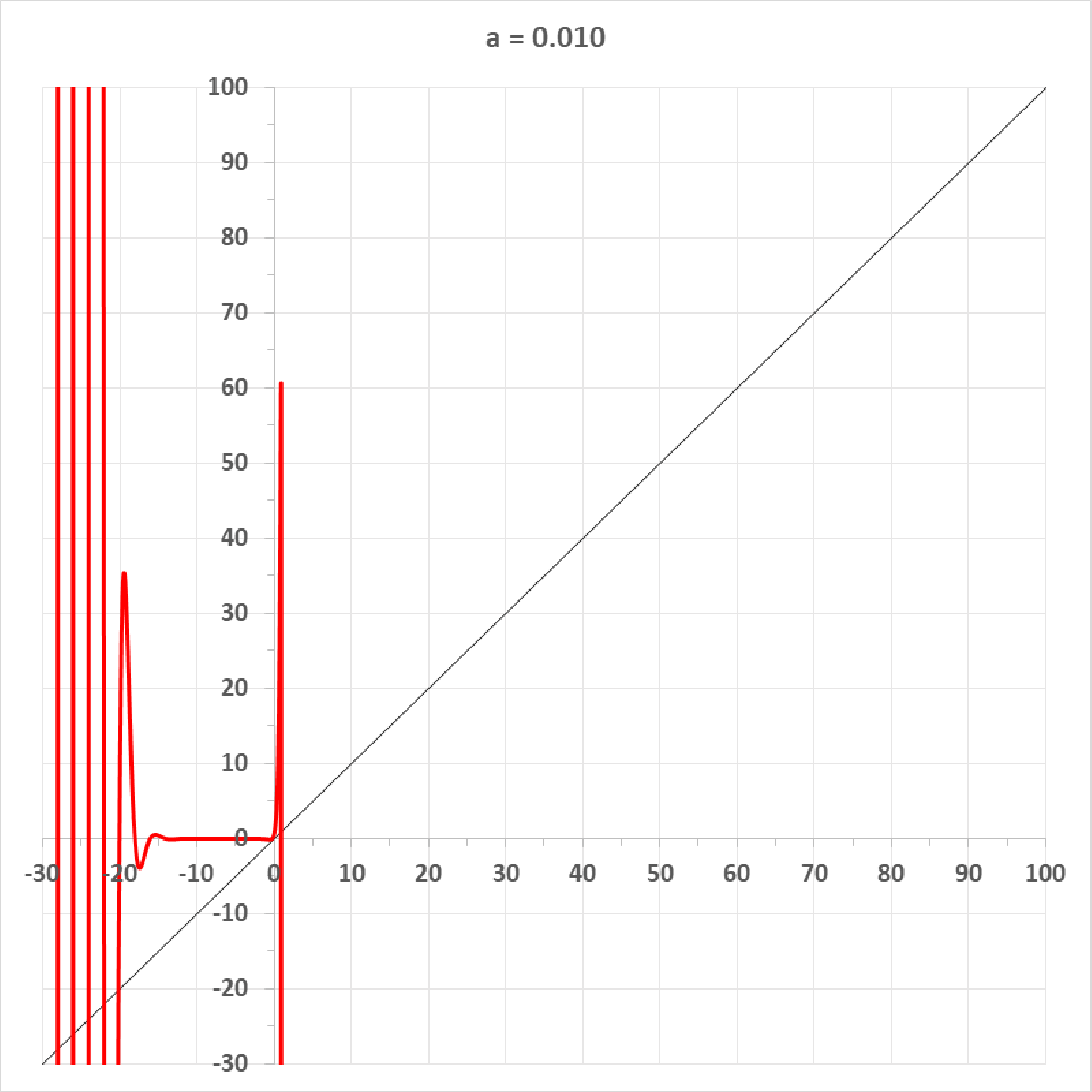
$$\zeta\left(1-s, \frac{m}{n}\right) = \frac{2\Gamma(s)}{(2\pi n)^s} \sum_{k=1}^{n} \left[\cos\left(\frac{\pi s}{2} - \frac{2\pi k m}{n}\right) \zeta\left(s, \frac{k}{n}\right)\right] \qquad m,n \in \mathbb{Z}, 1 \leq m \leq n \tag{2}$$

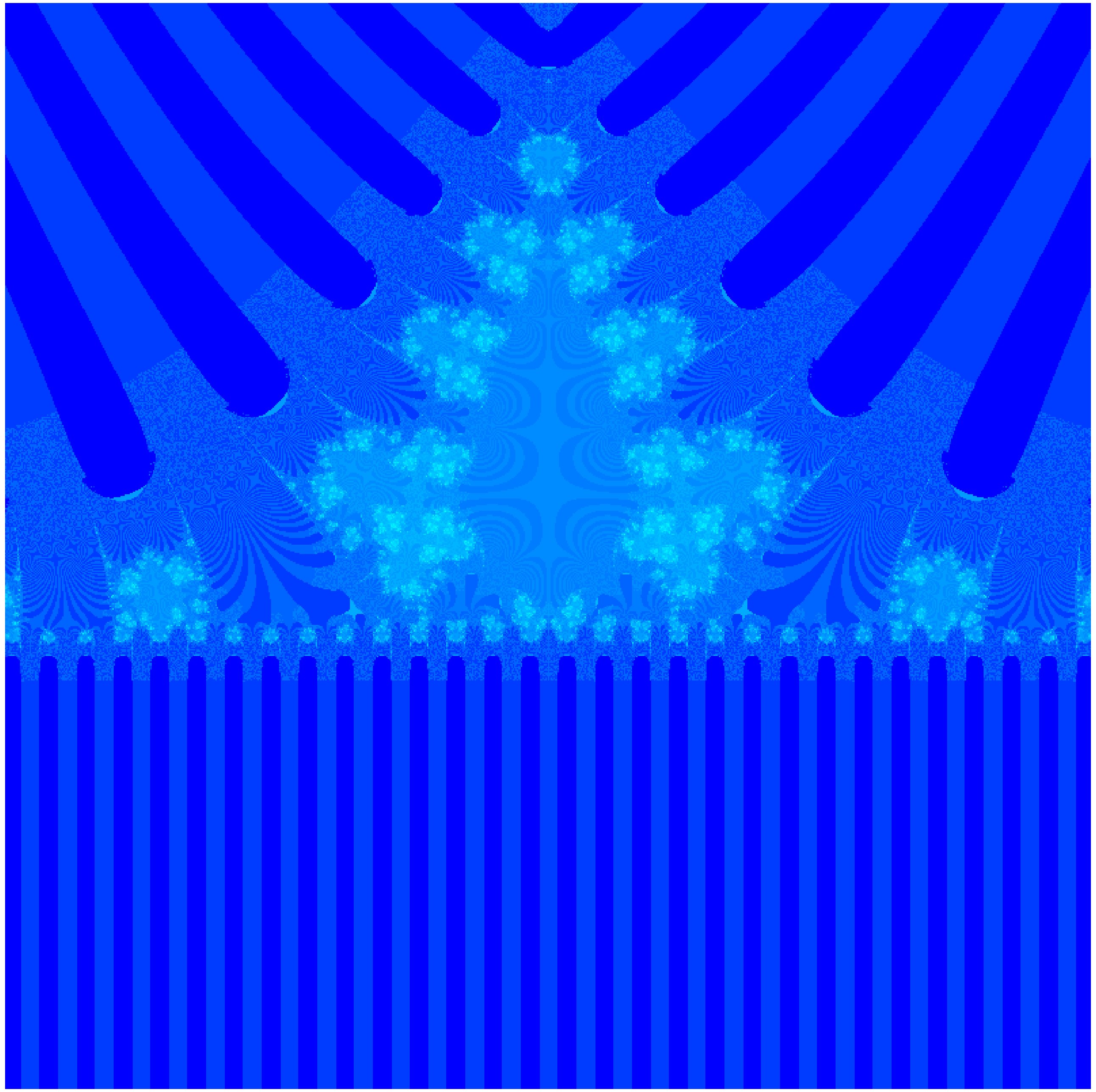
We need only consider $0 < a \leq 1$ because:

$$\zeta(s, a+m) = \zeta(s,a) - \sum_{n=0}^{m-1} \frac{1}{(n+a)^s} \qquad m \in \mathbb{Z} \tag{3}$$

Approximations of Hurwitz zeta functions by `zeta_machine` relate to $a \in \mathbb{Q}$ only. Precision is limited to about 12 significant figures.

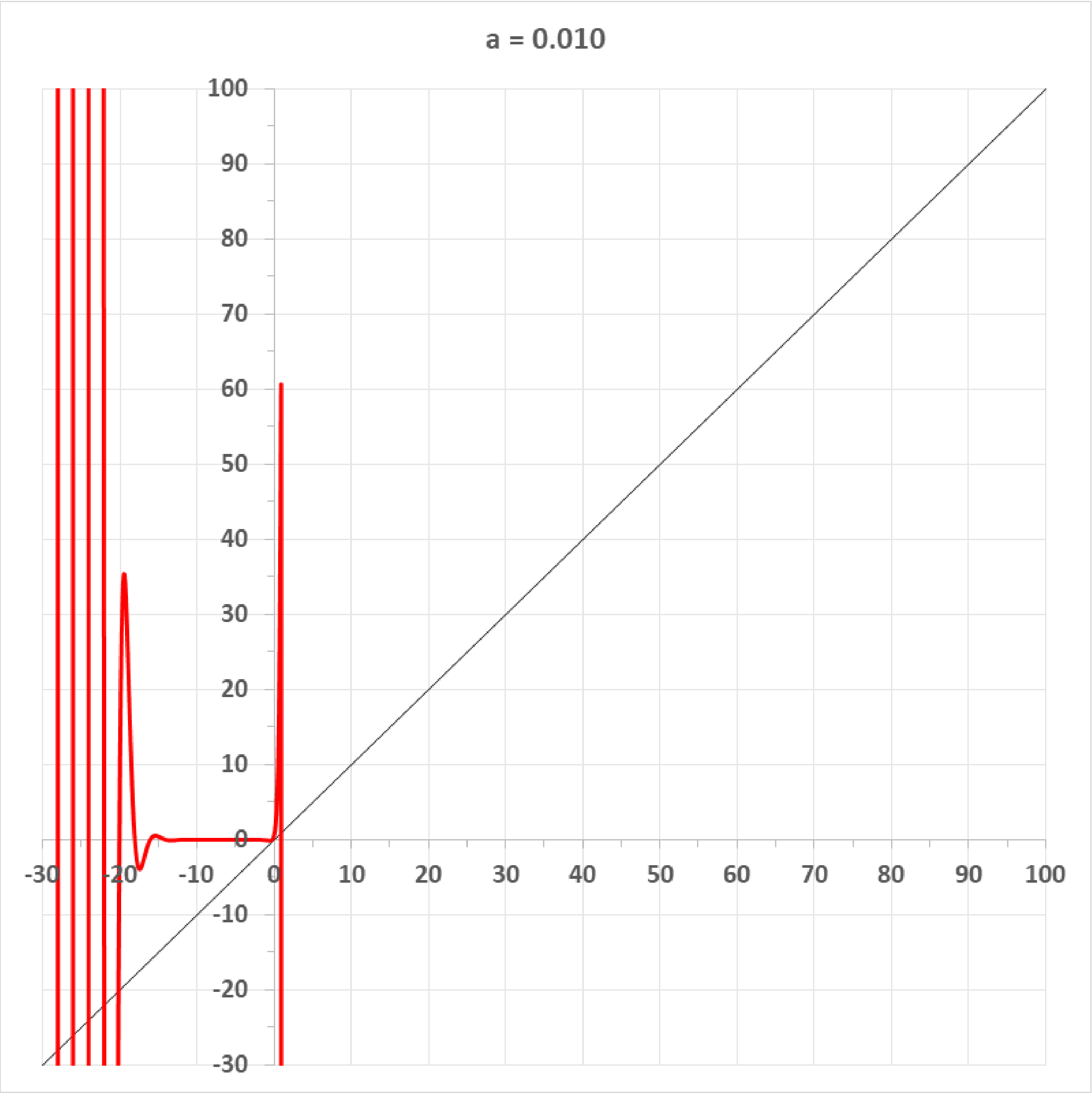
Hurwitz zeta functions

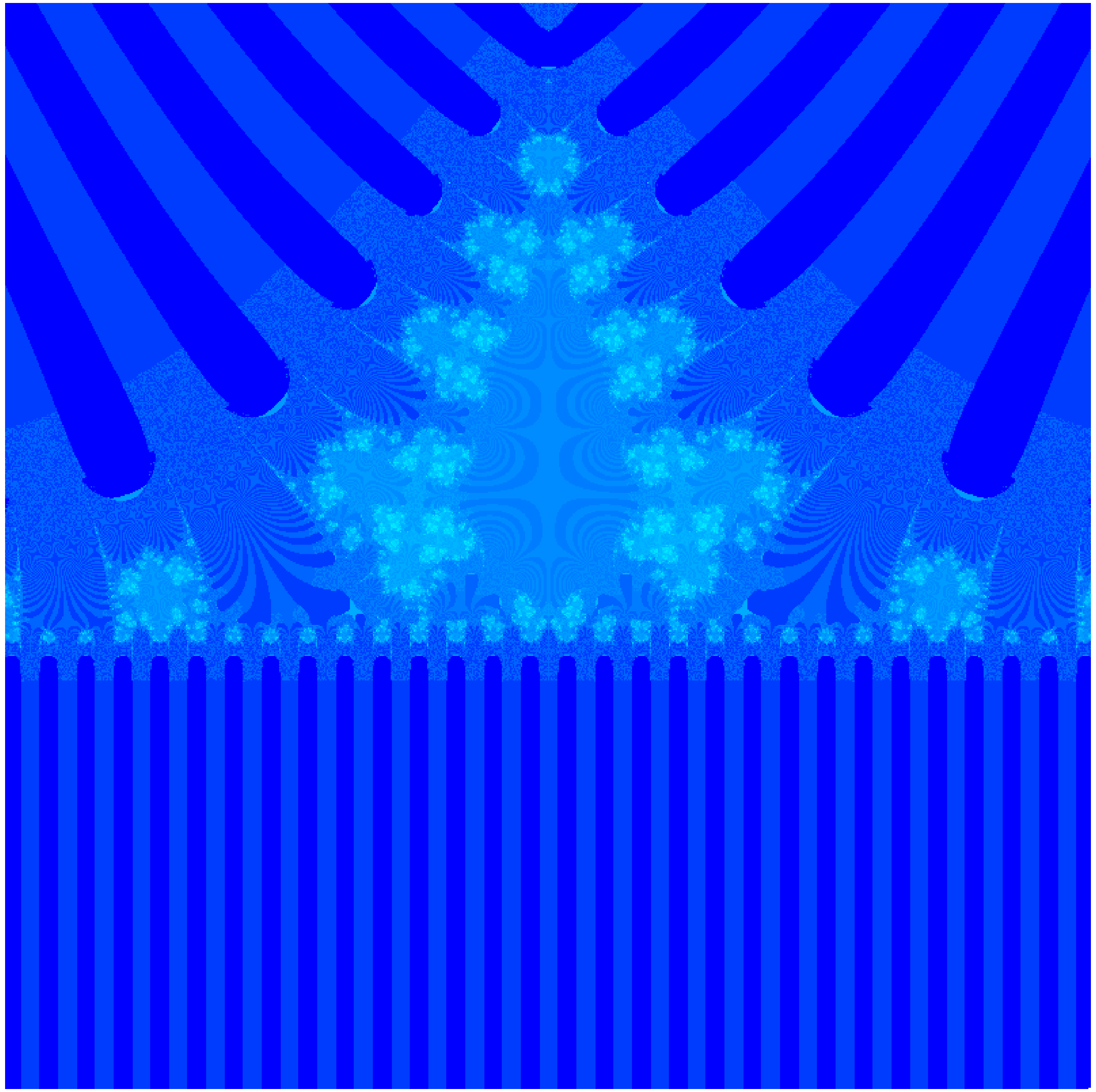




$-23 \leq \Re(s) \leq 31, -20 \leq \Im(s) \leq 20, 20ppu$

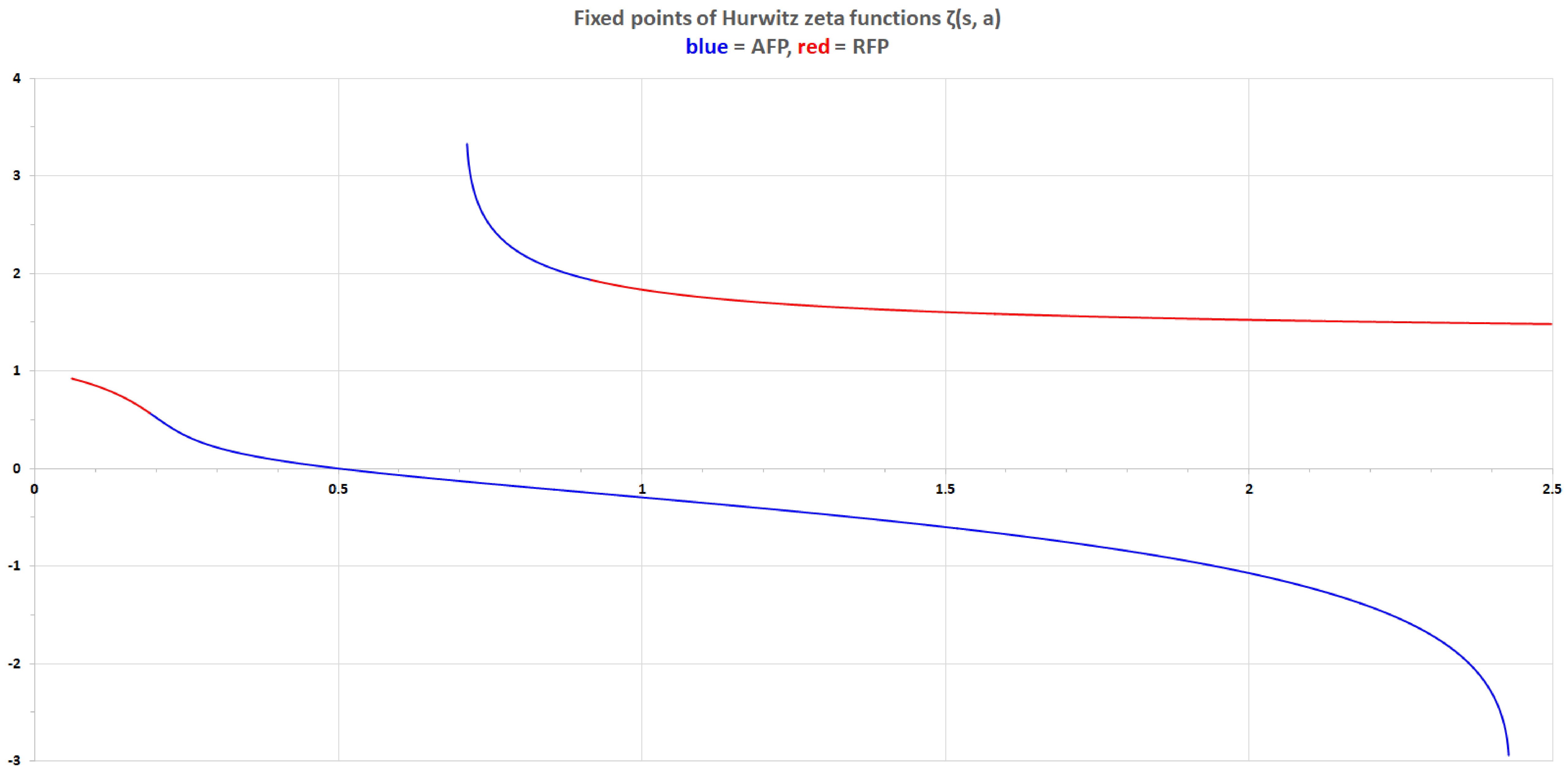
Hurwitz zeta functions



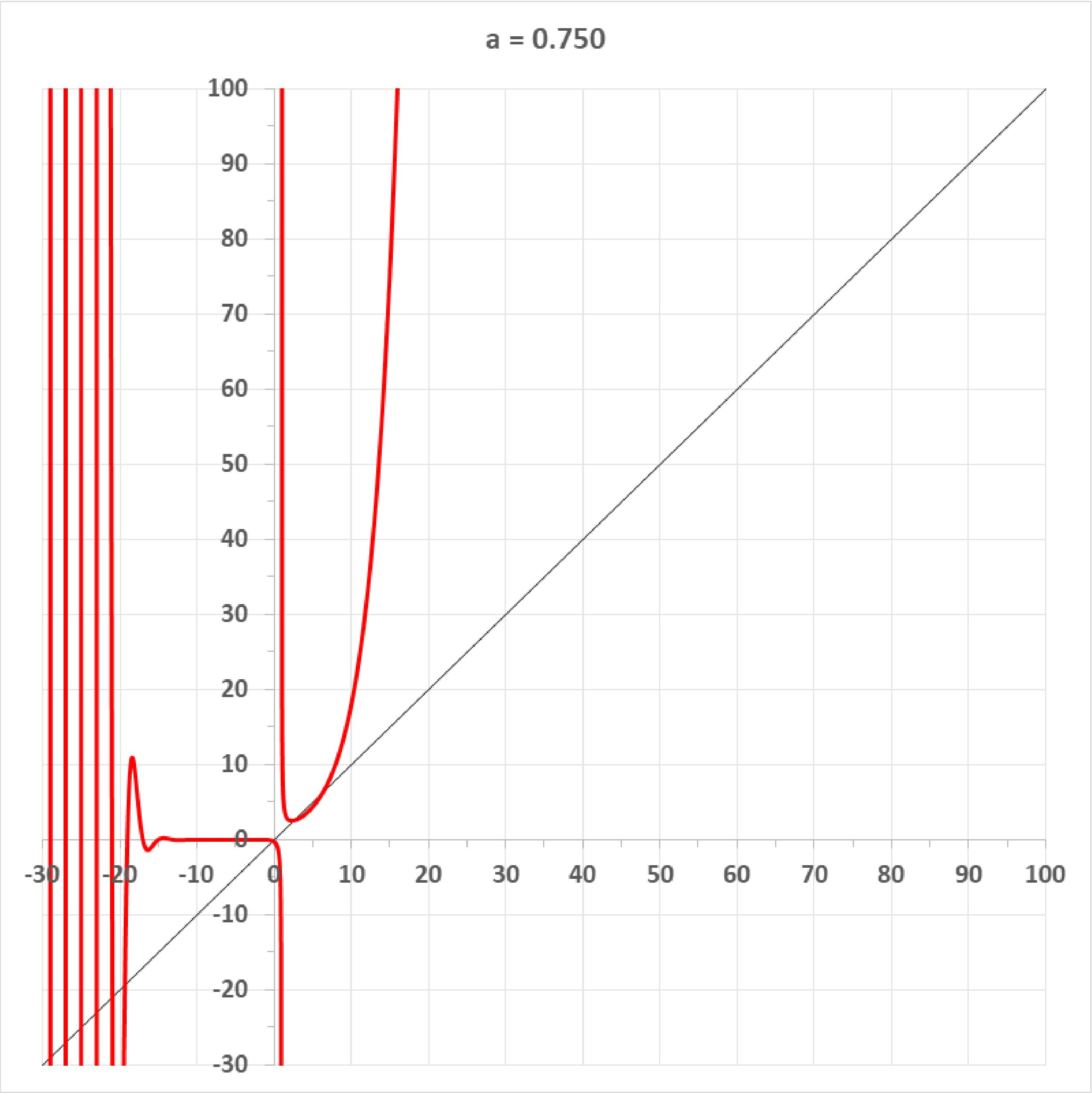


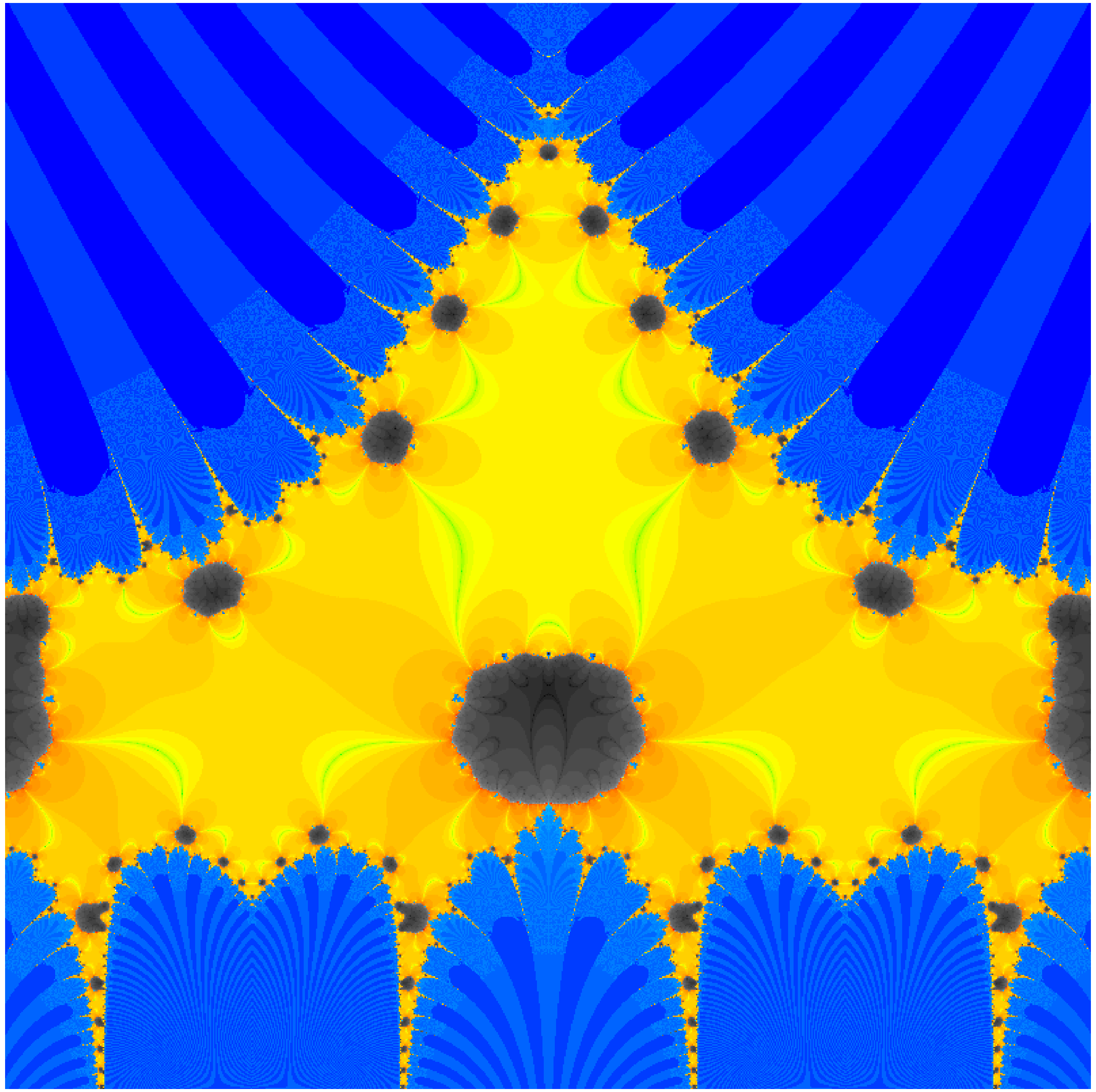
$-23 \leq \Re(s) \leq 31, -20 \leq \Im(s) \leq 20, 20ppu$

Hurwitz zeta functions



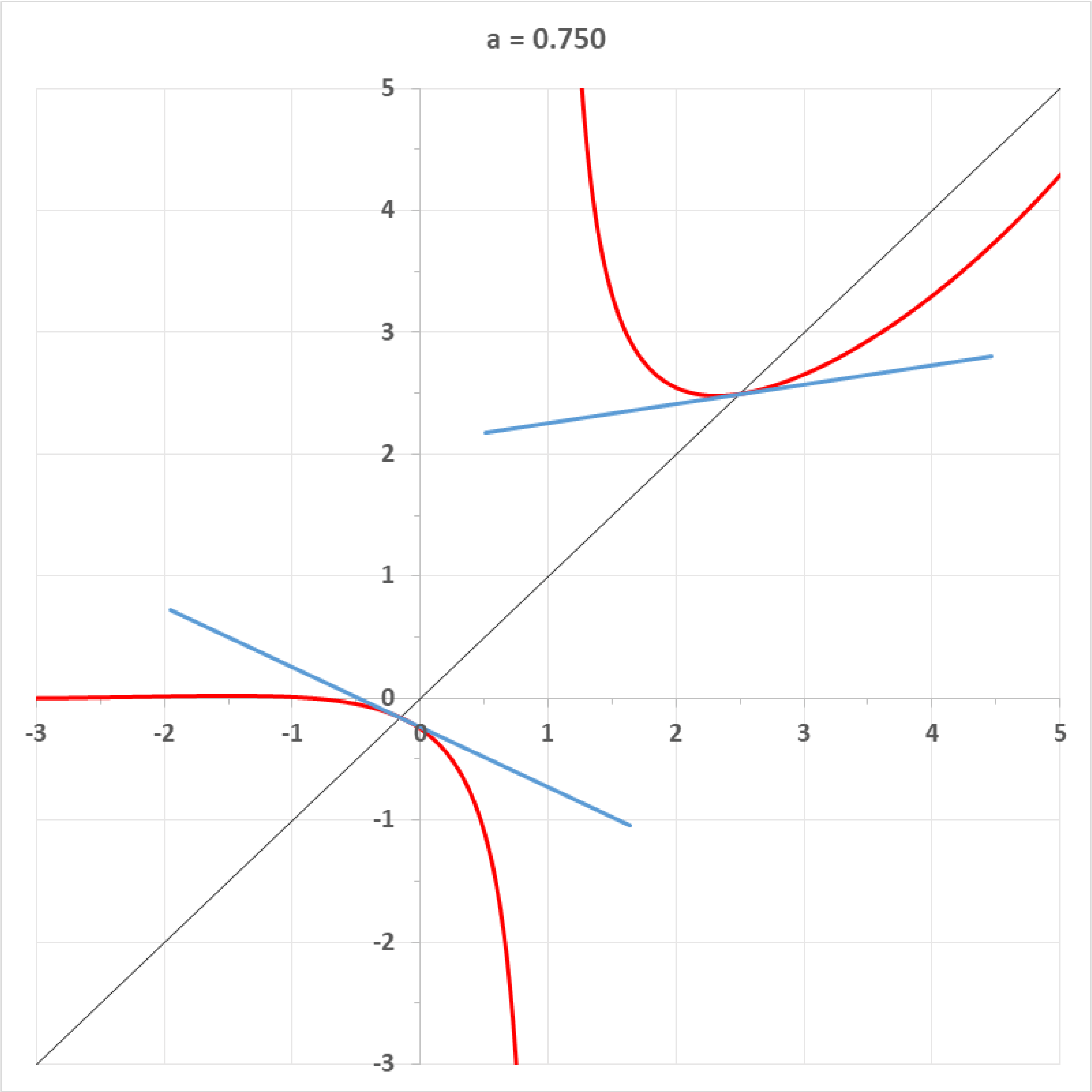
Hurwitz zeta functions

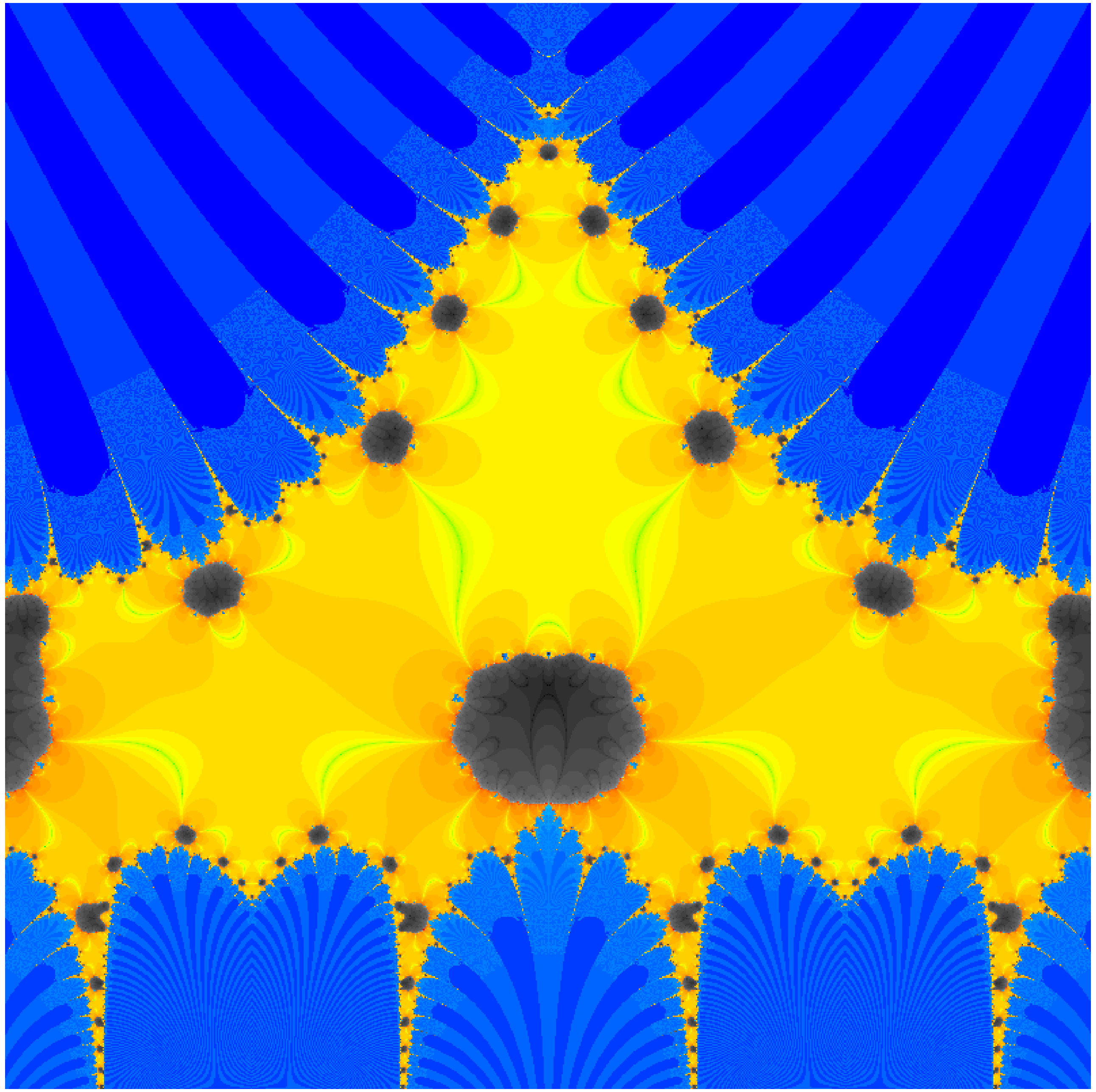




$-23 \leq \Re(s) \leq 31, -20 \leq \Im(s) \leq 20, 20ppu$

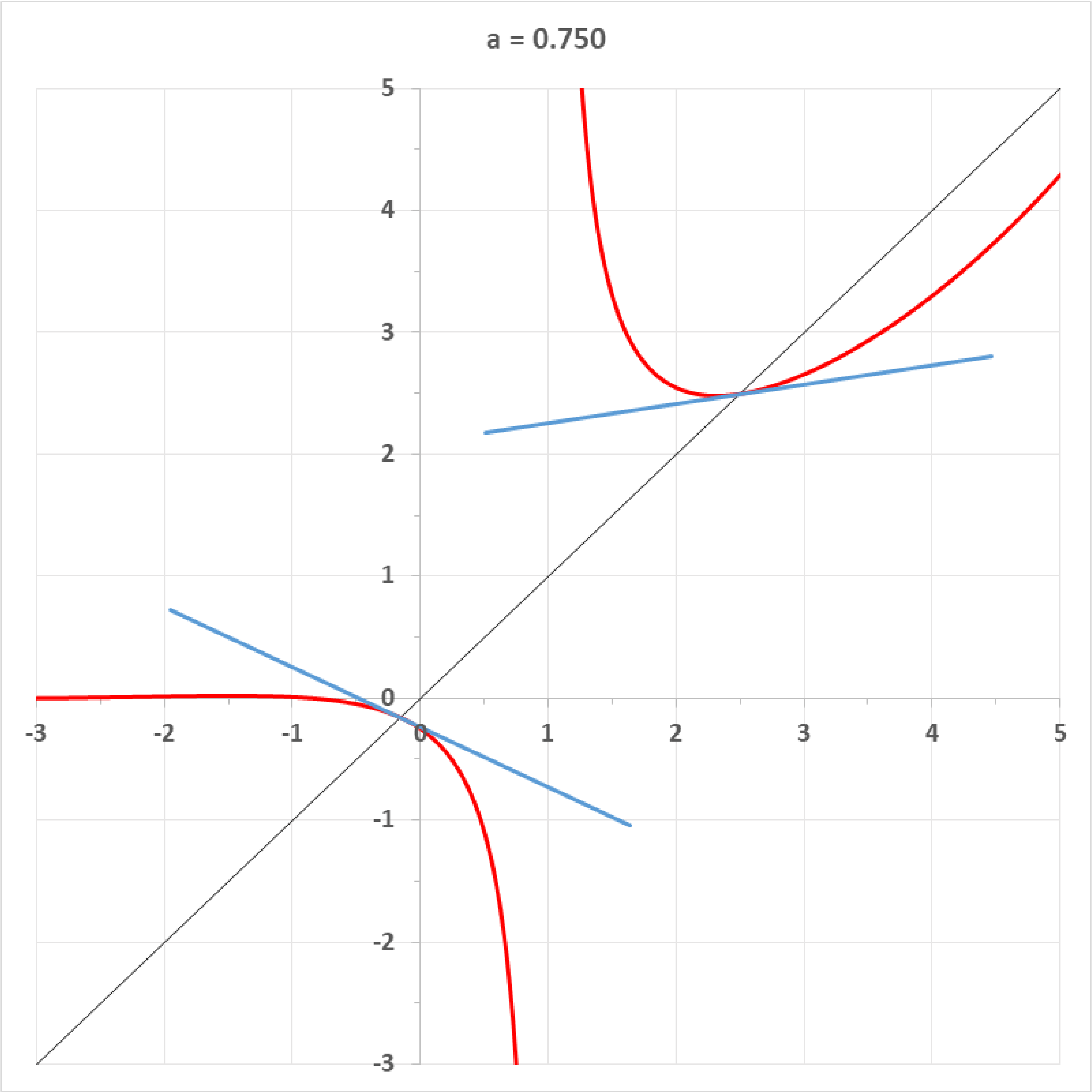
Hurwitz zeta functions

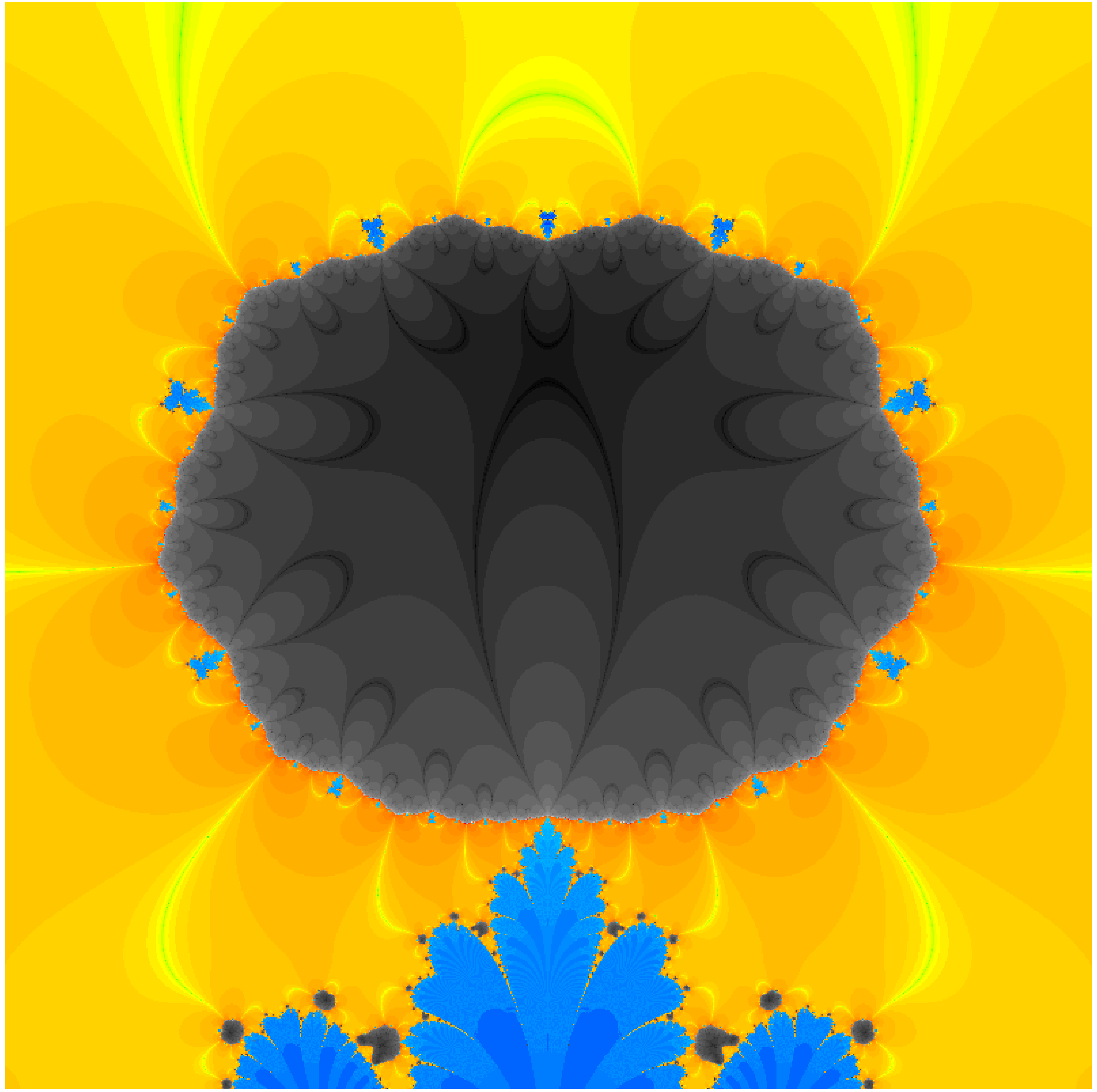




$-23 \leq \Re(s) \leq 31, -20 \leq \Im(s) \leq 20, 20ppu$

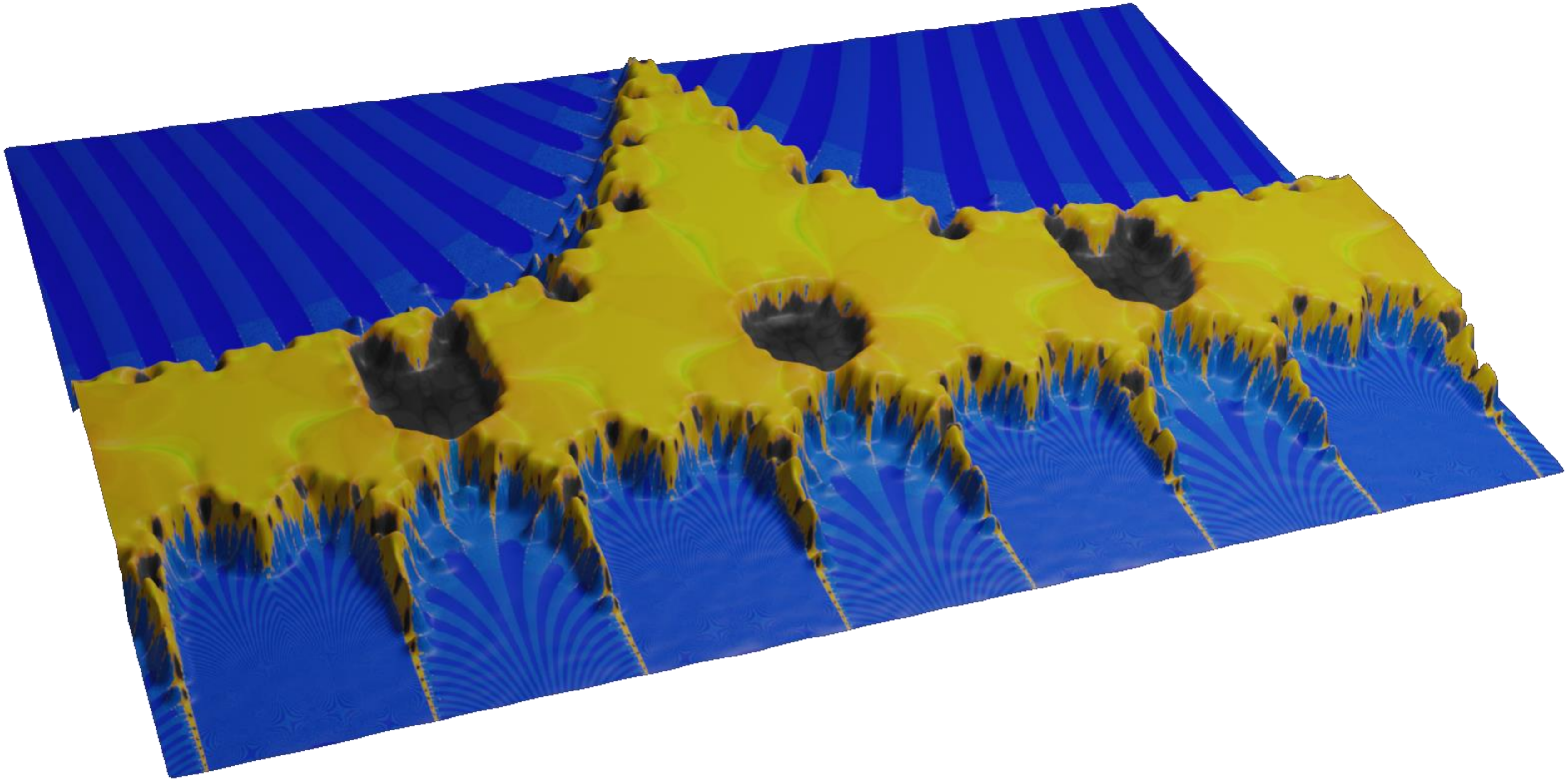
Hurwitz zeta functions



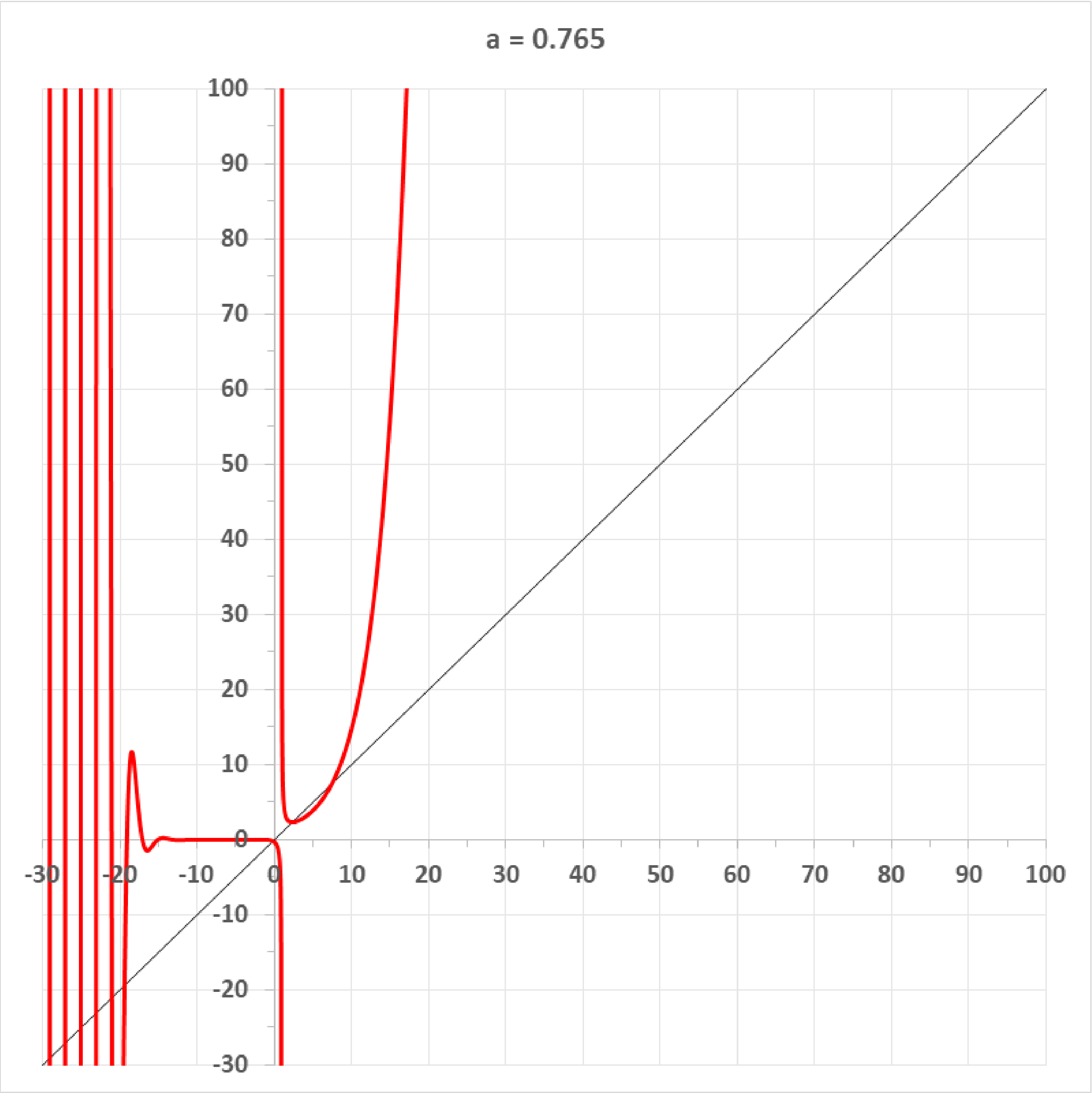


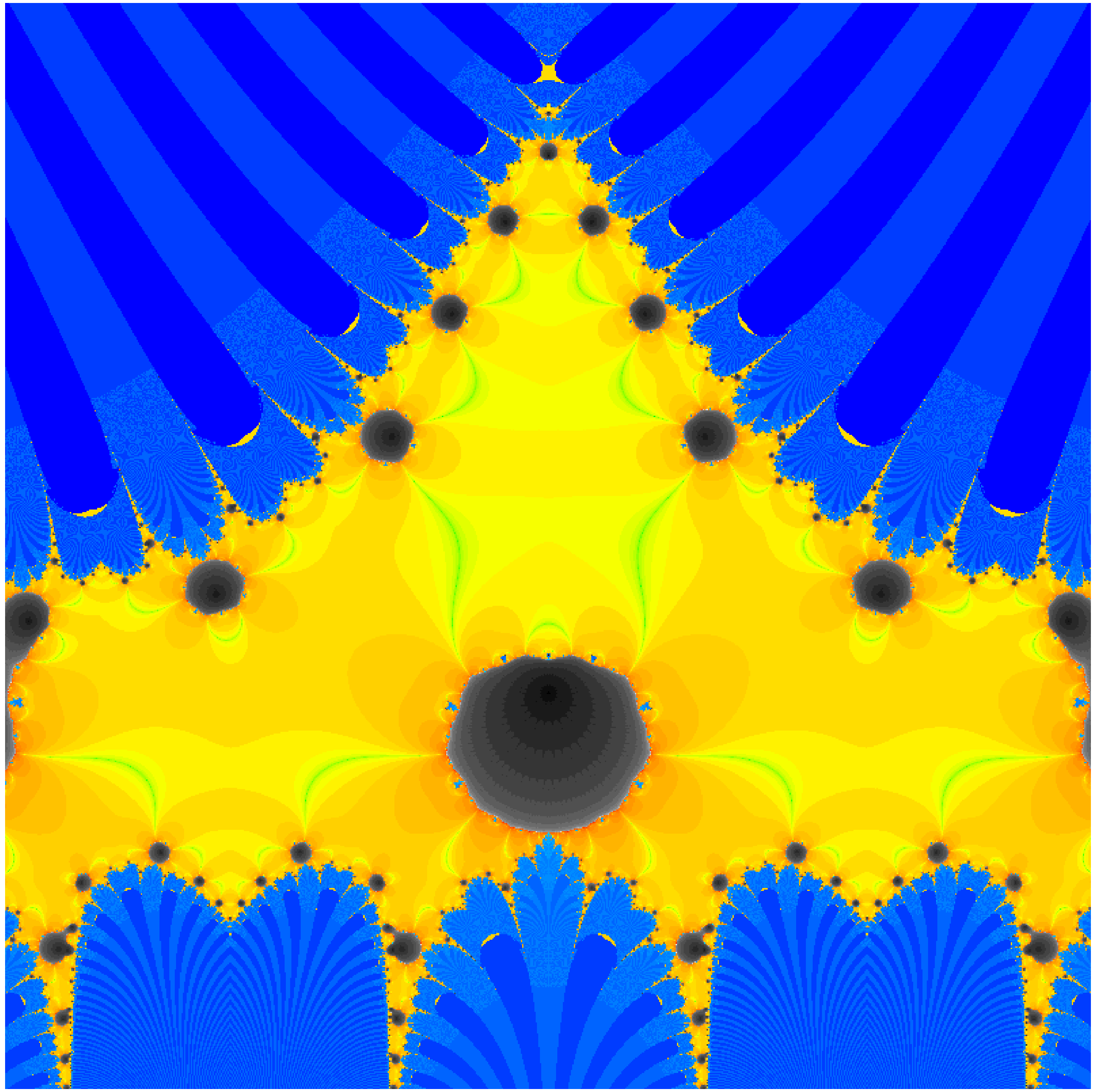
$-1 \leq \Re(s) \leq 9, -5 \leq \Im(s) \leq 5, 80ppu$

Hurwitz zeta functions



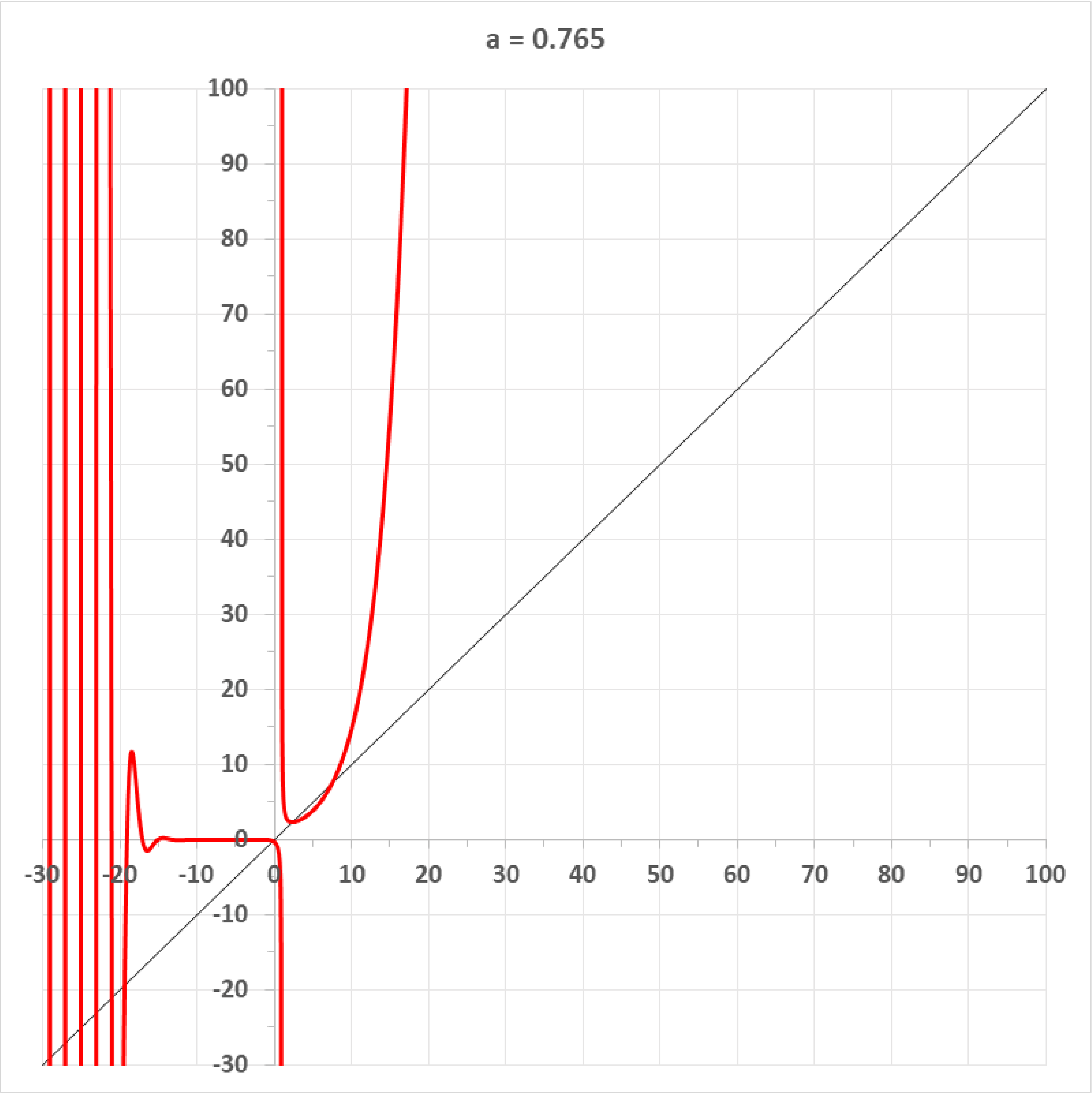
Hurwitz zeta functions

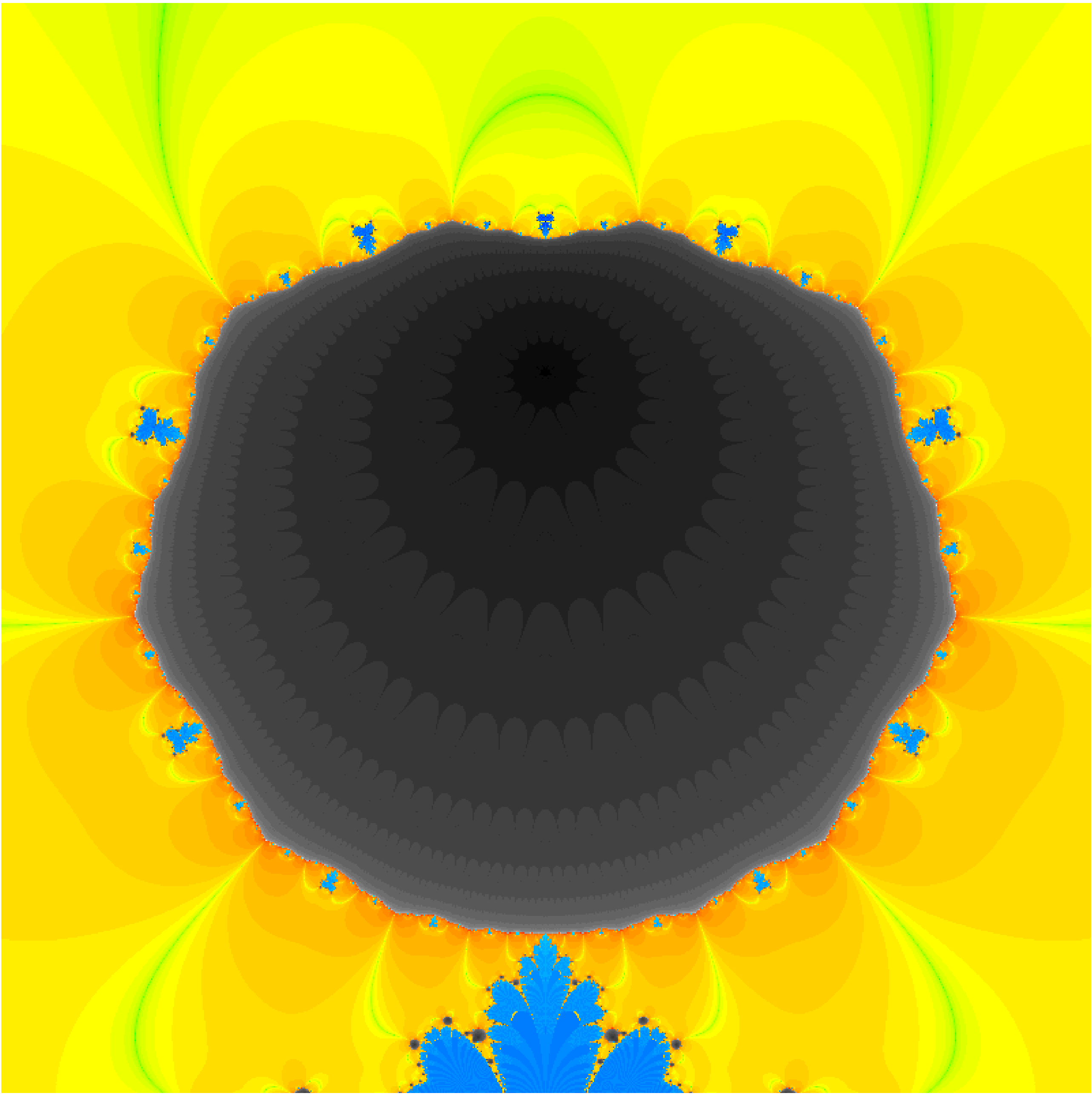




$-23 \leq \Re(s) \leq 31, -20 \leq \Im(s) \leq 20, 20ppu$

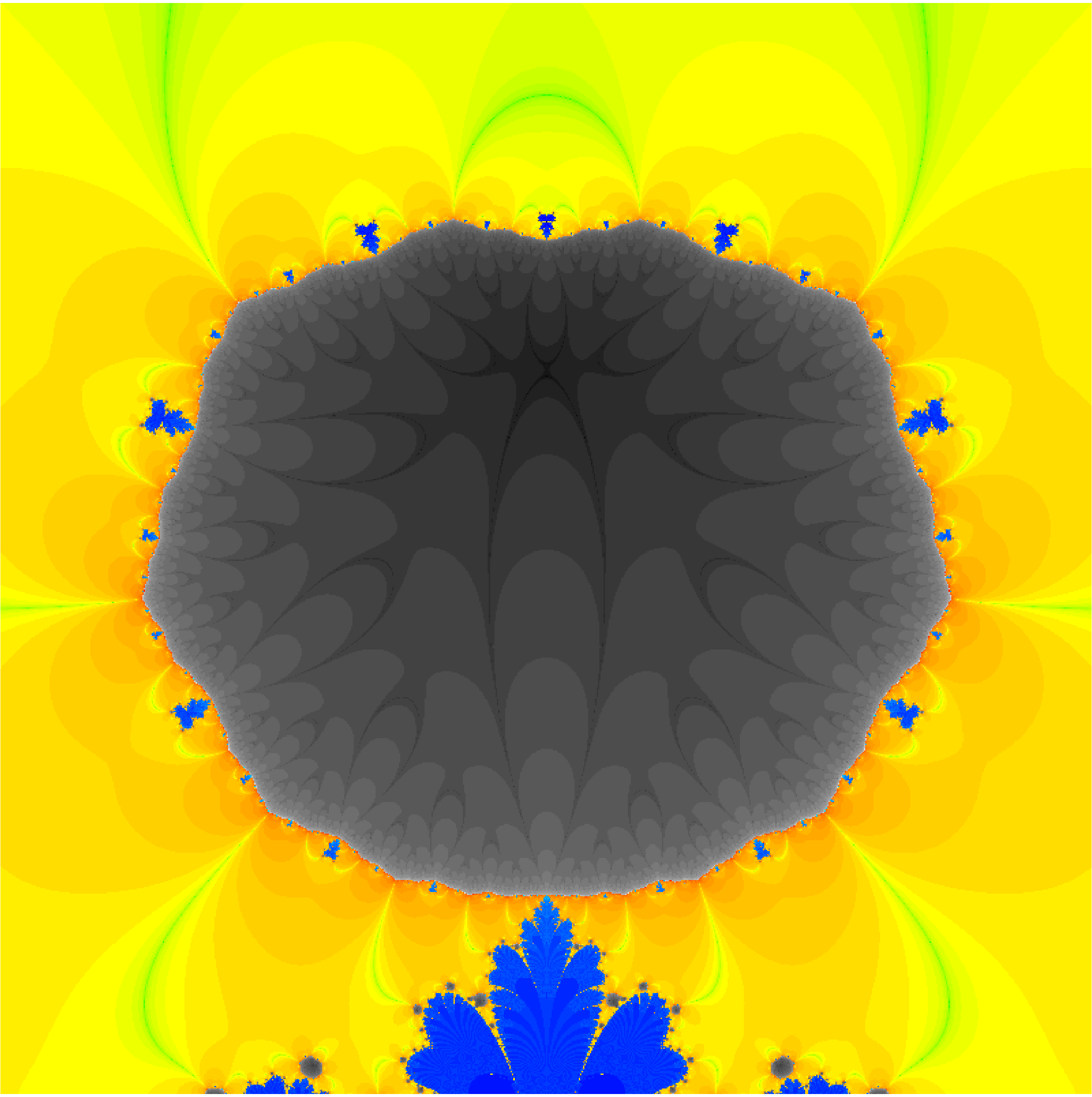
Hurwitz zeta functions



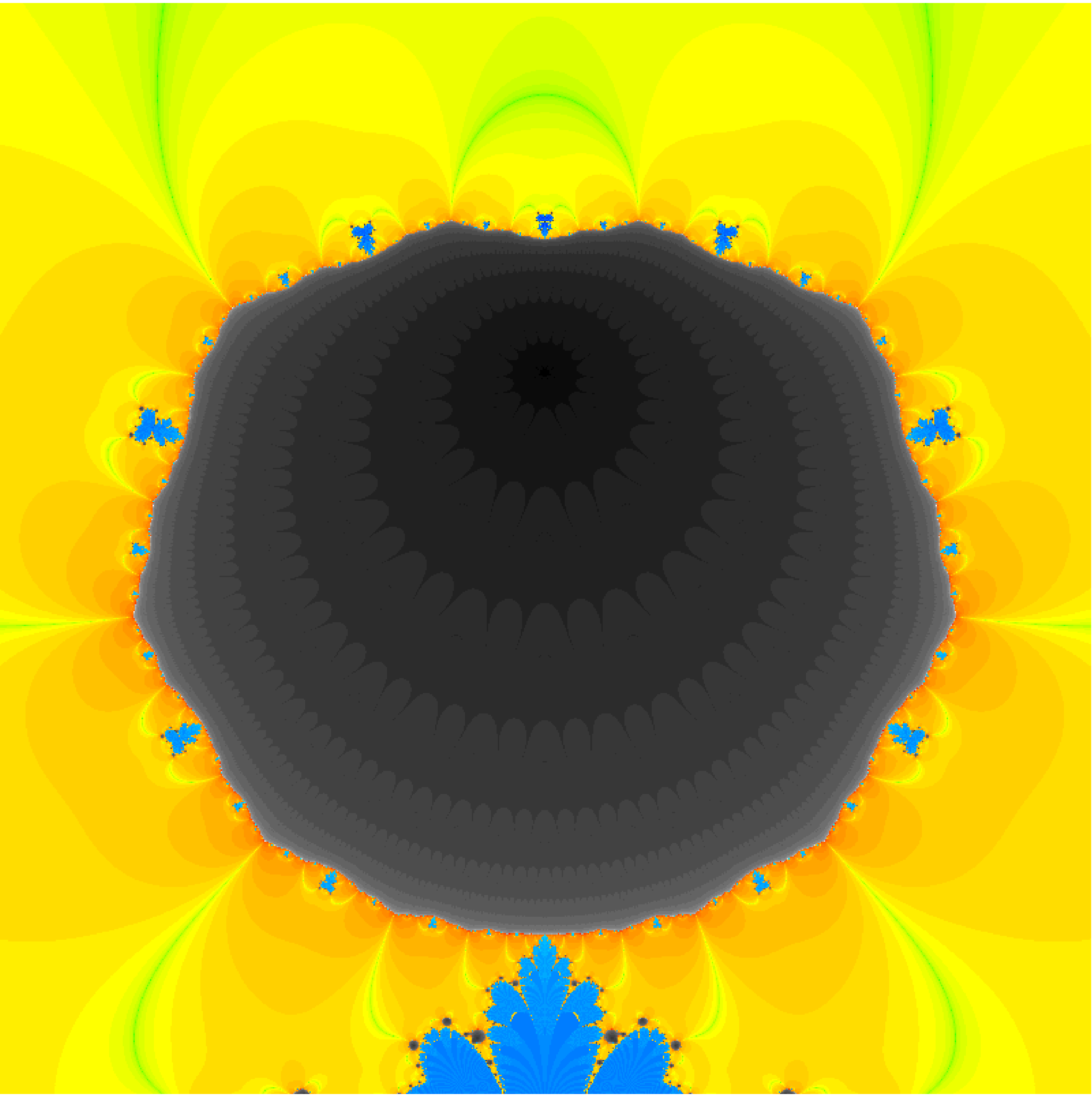


$-1 \leq \Re(s) \leq 9, -5 \leq \Im(s) \leq 5, 80ppu$

Hurwitz zeta functions

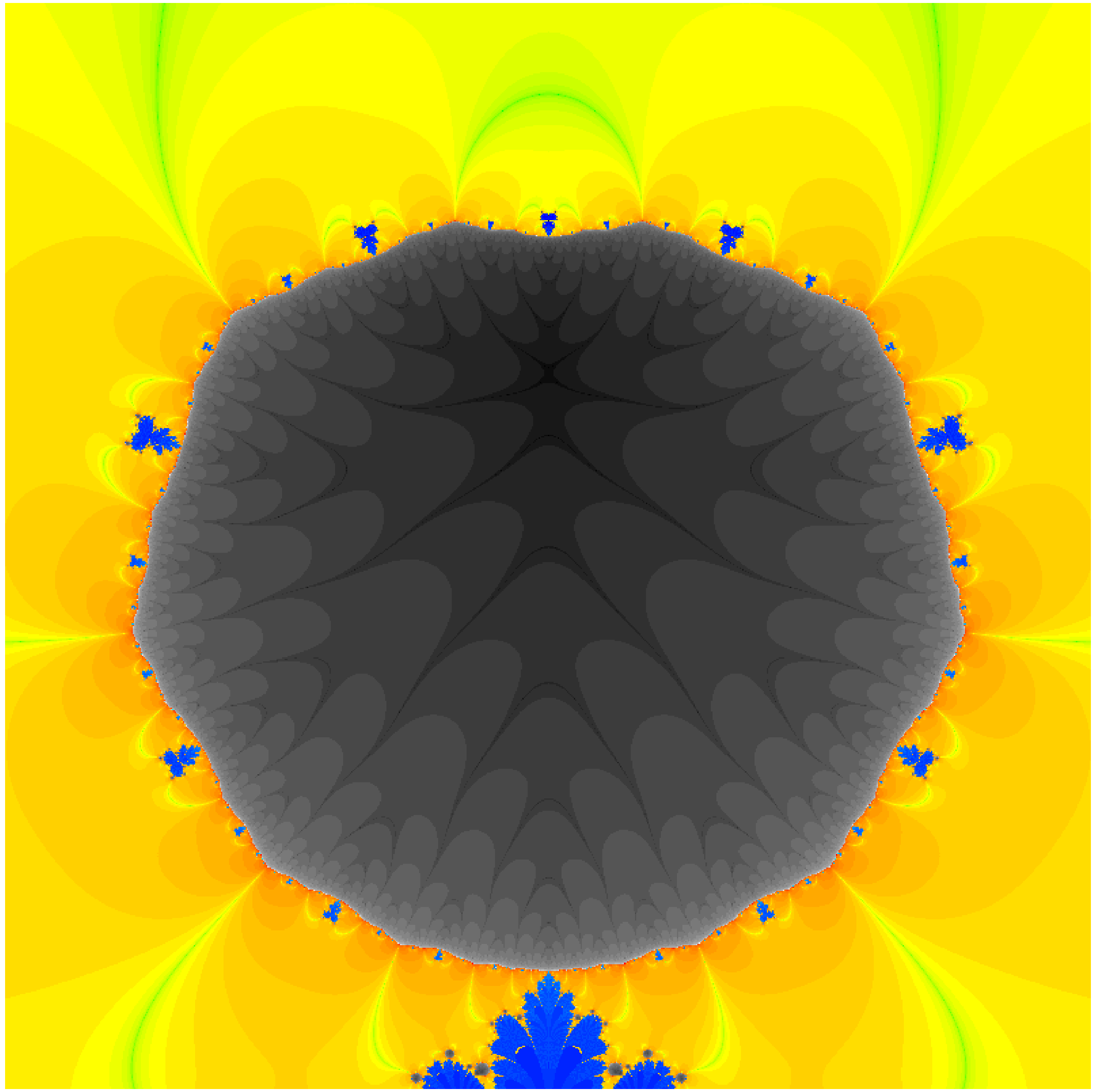


$-1 \leq \Re(s) \leq 9, -5 \leq \Im(s) \leq 5, 80ppu$

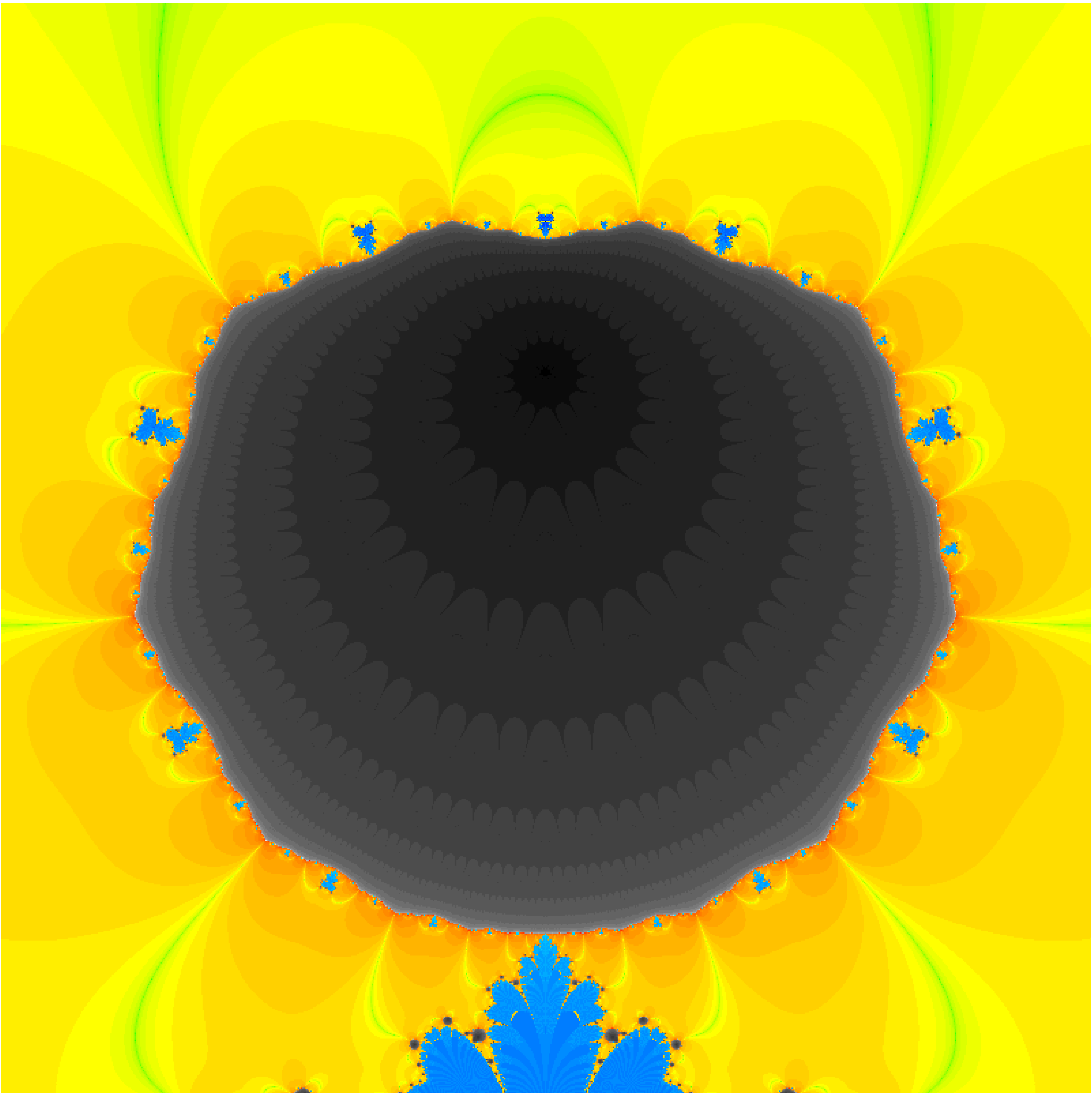


$-1 \leq \Re(s) \leq 9, -5 \leq \Im(s) \leq 5, 80ppu$

Hurwitz zeta functions

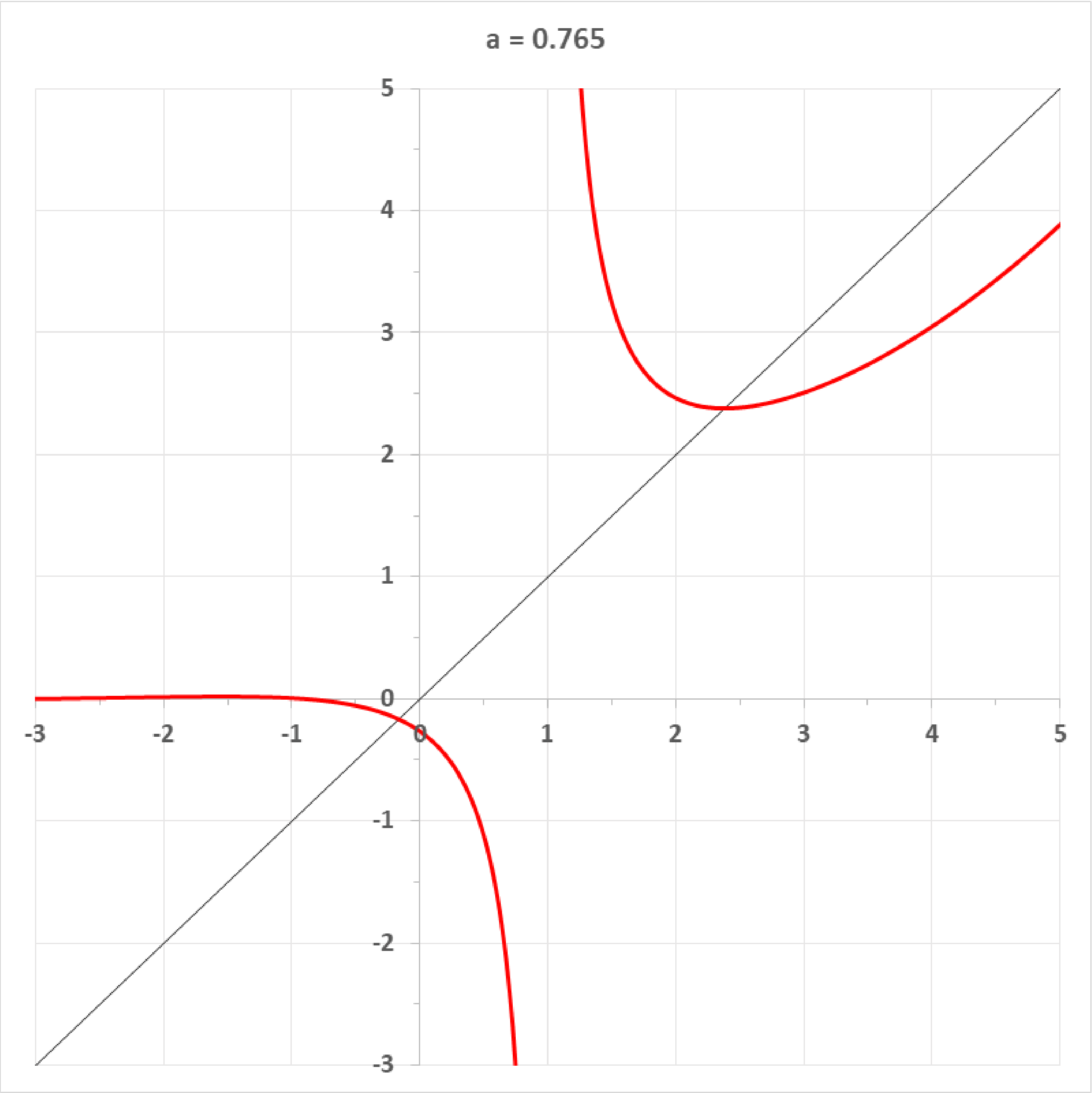


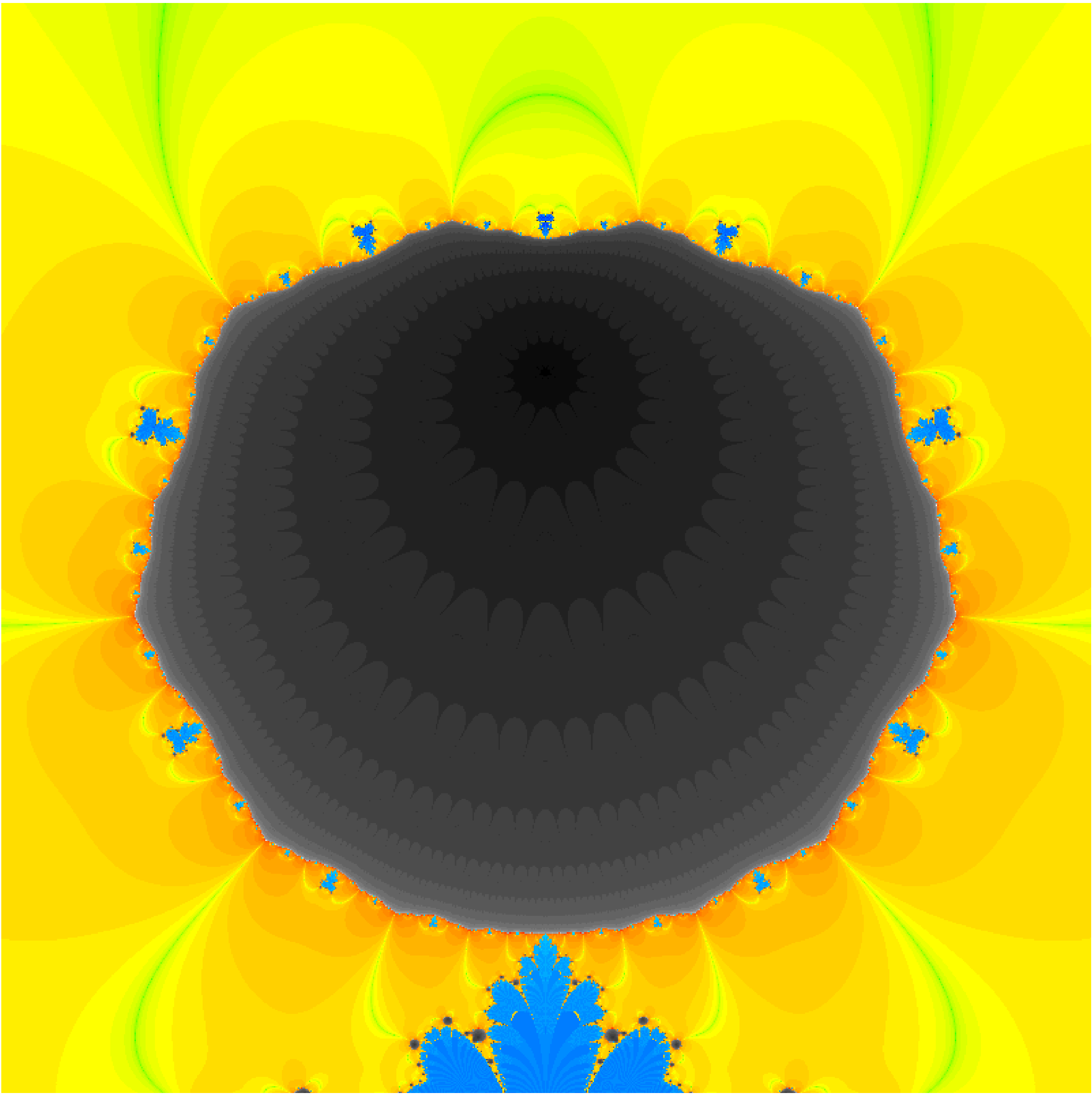
$-1 \leq \Re(s) \leq 9, -5 \leq \Im(s) \leq 5, 80ppu$



$-1 \leq \Re(s) \leq 9, -5 \leq \Im(s) \leq 5, 80ppu$

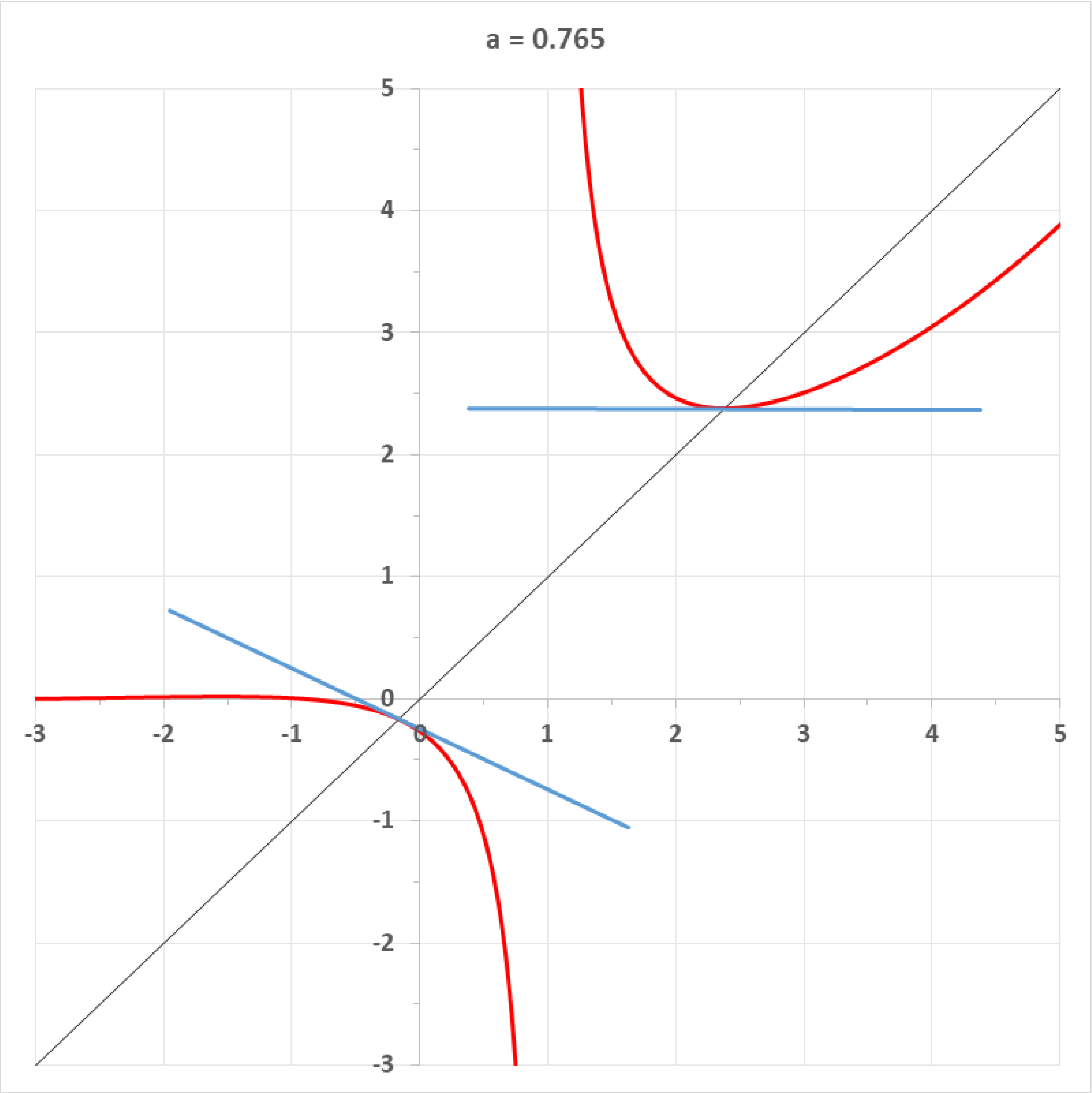
Hurwitz zeta functions

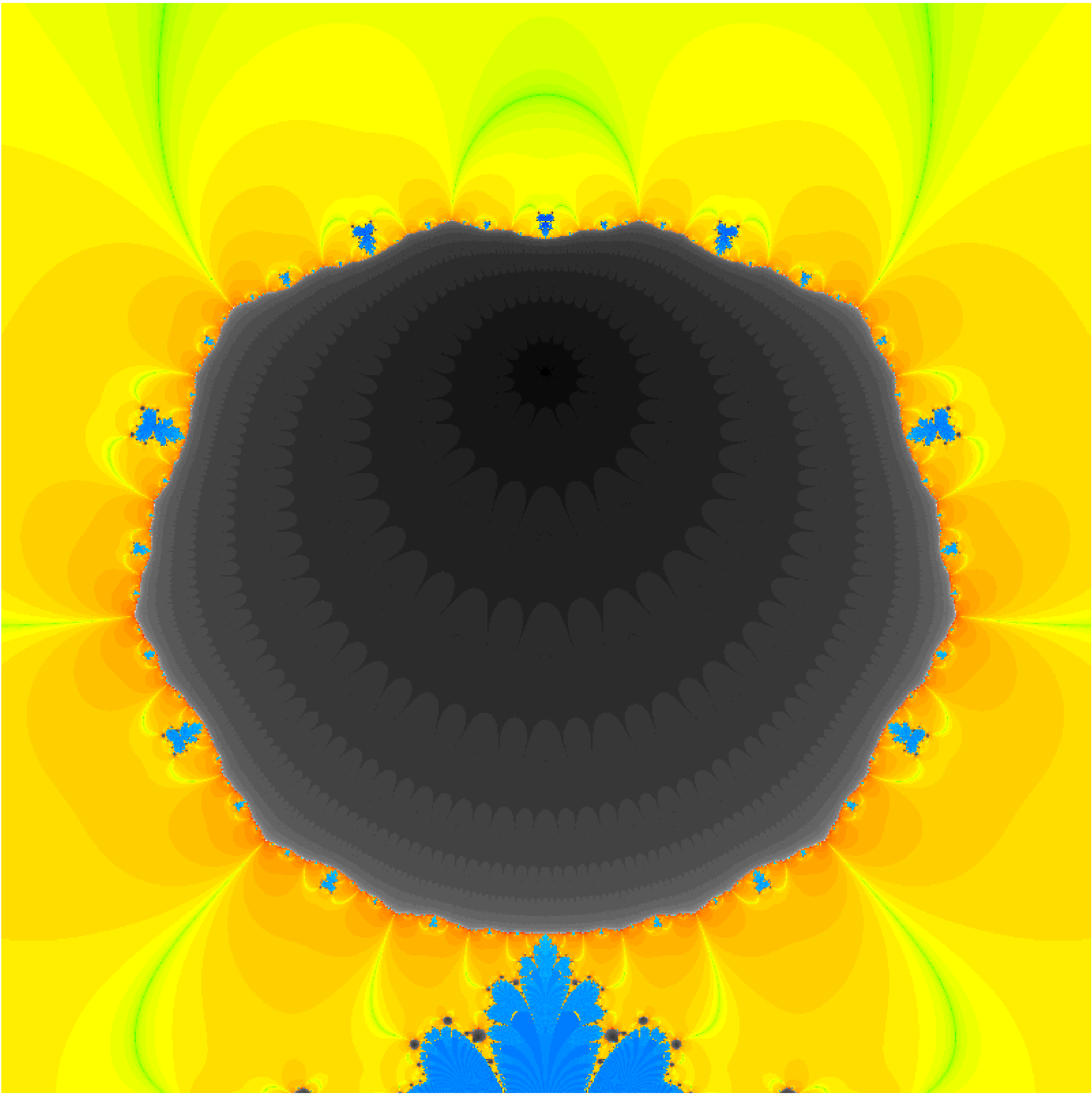




$-1 \leq \Re(s) \leq 9, -5 \leq \Im(s) \leq 5, 80ppu$

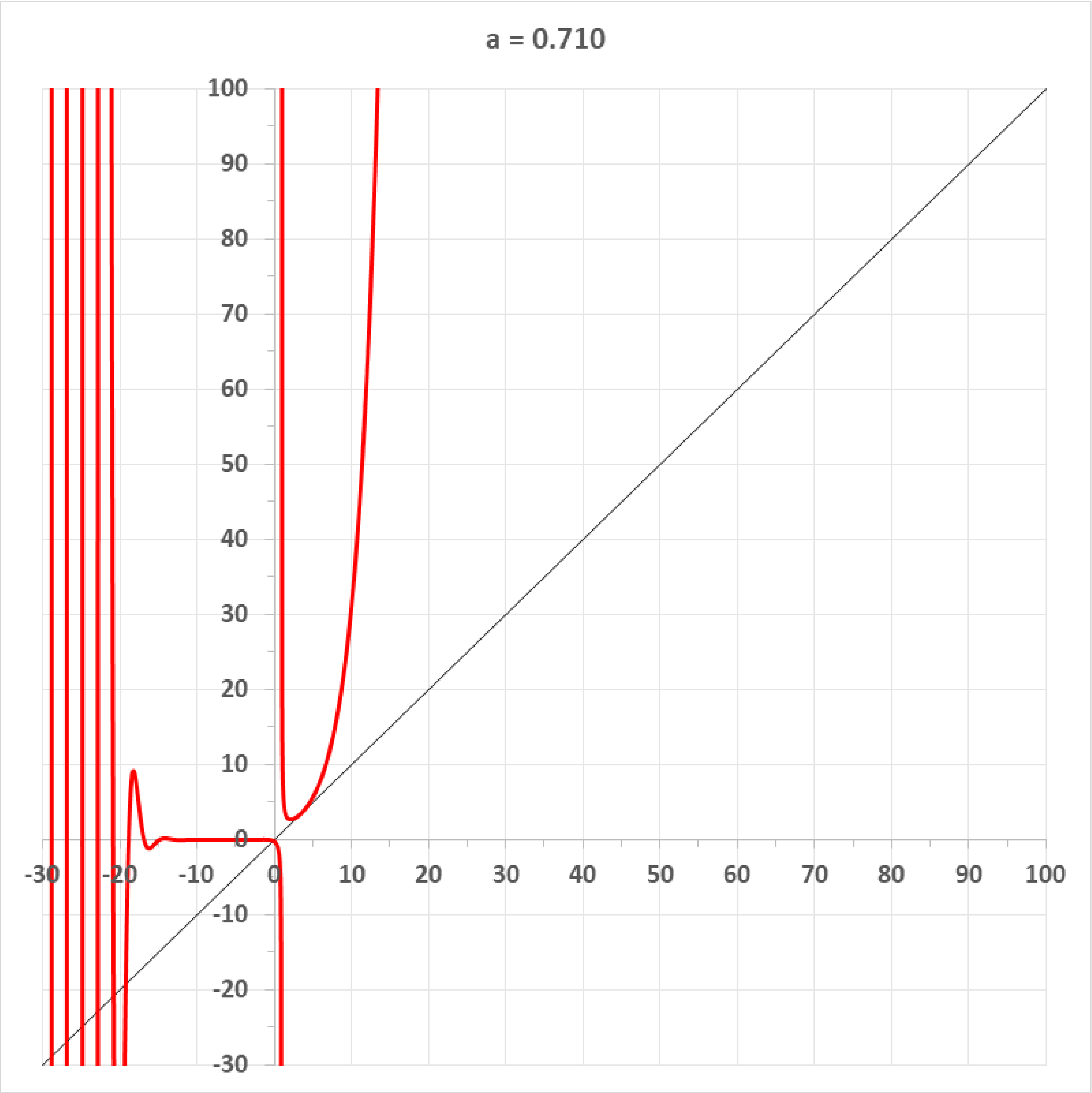
Hurwitz zeta functions

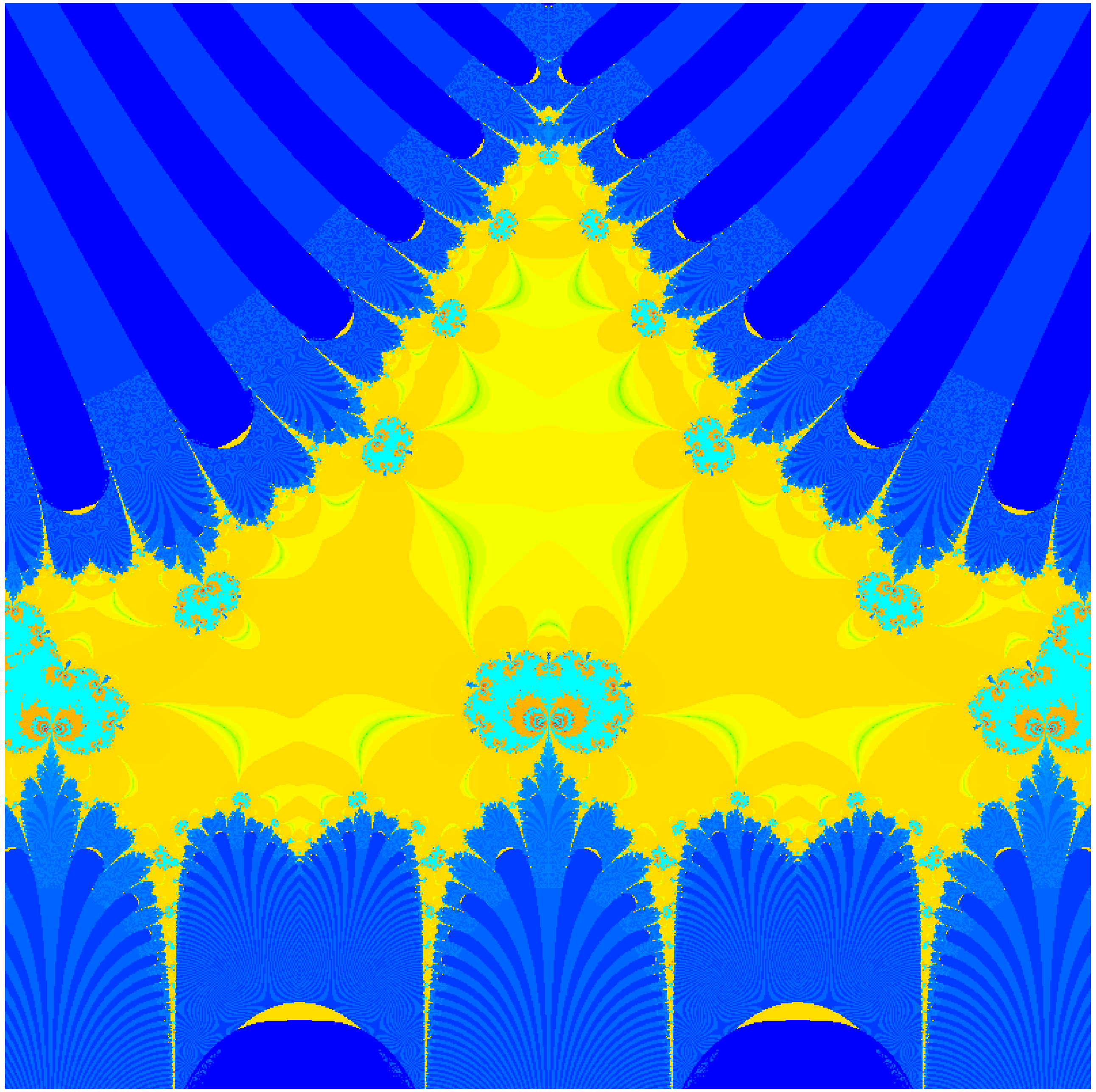




$-1 \leq \Re(s) \leq 9, -5 \leq \Im(s) \leq 5, 80ppu$

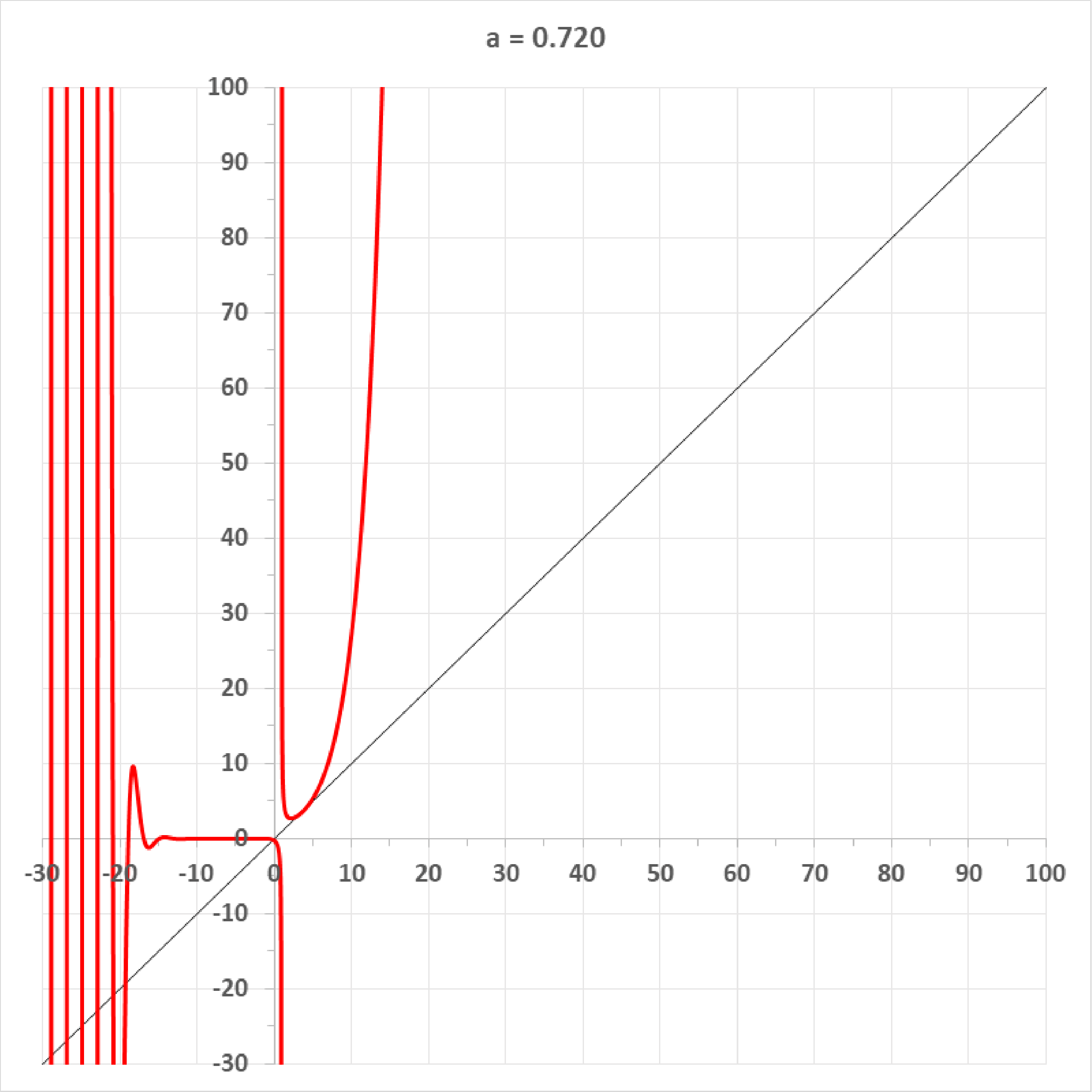
Hurwitz zeta functions

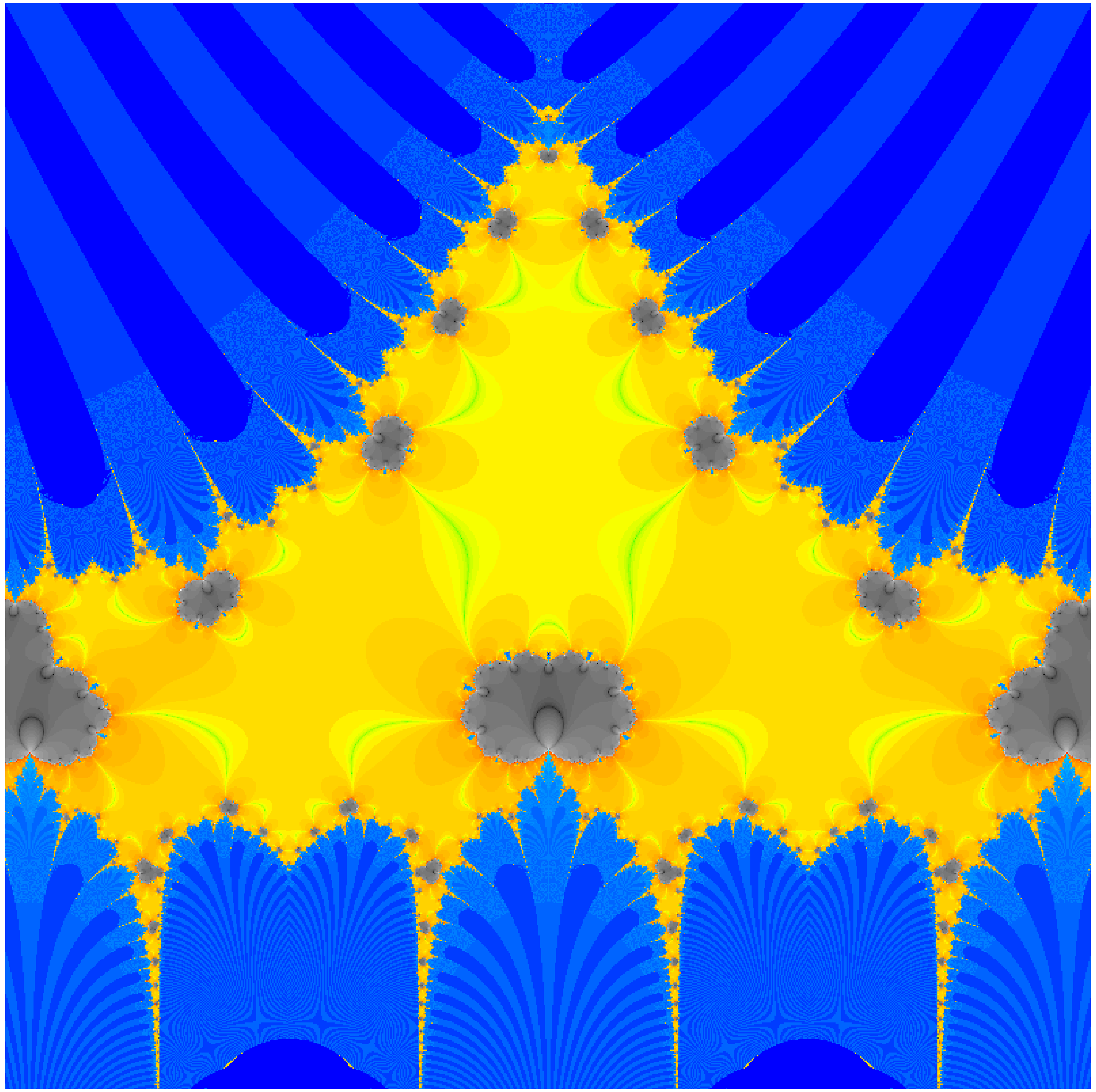




$-23 \leq \Re(s) \leq 31, -20 \leq \Im(s) \leq 20, 20ppu$

Hurwitz zeta functions





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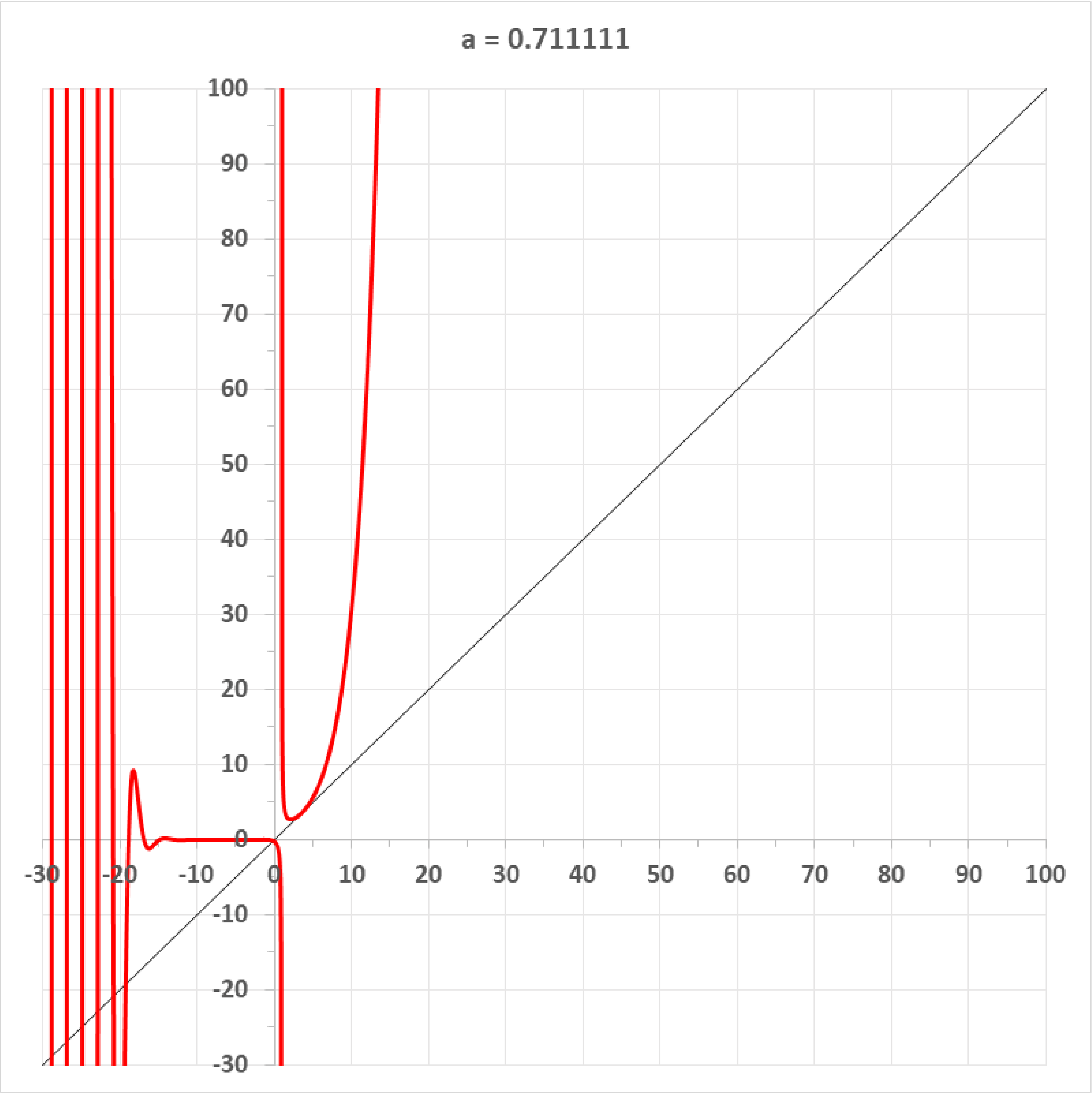
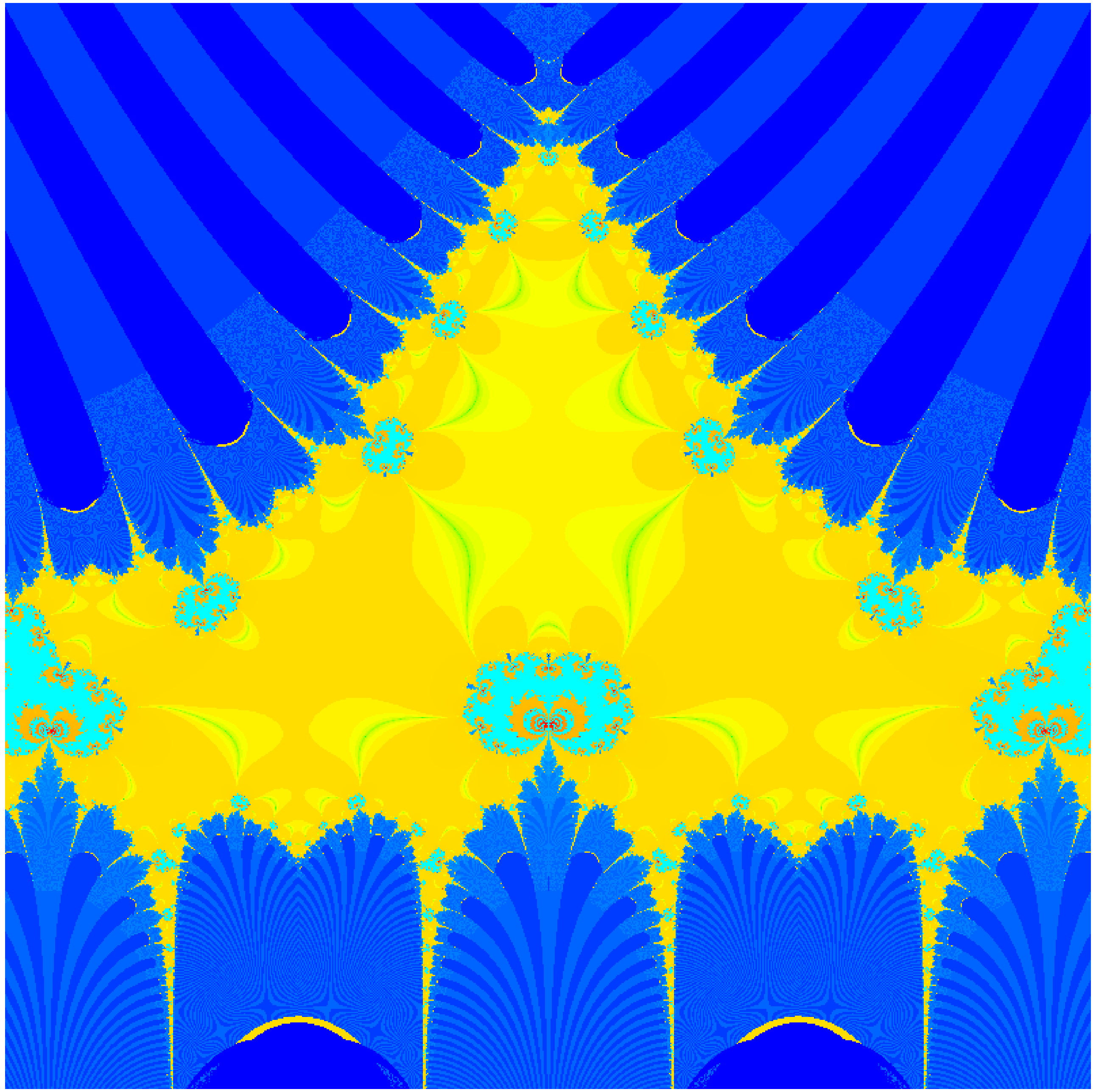
Hurwitz zeta functions





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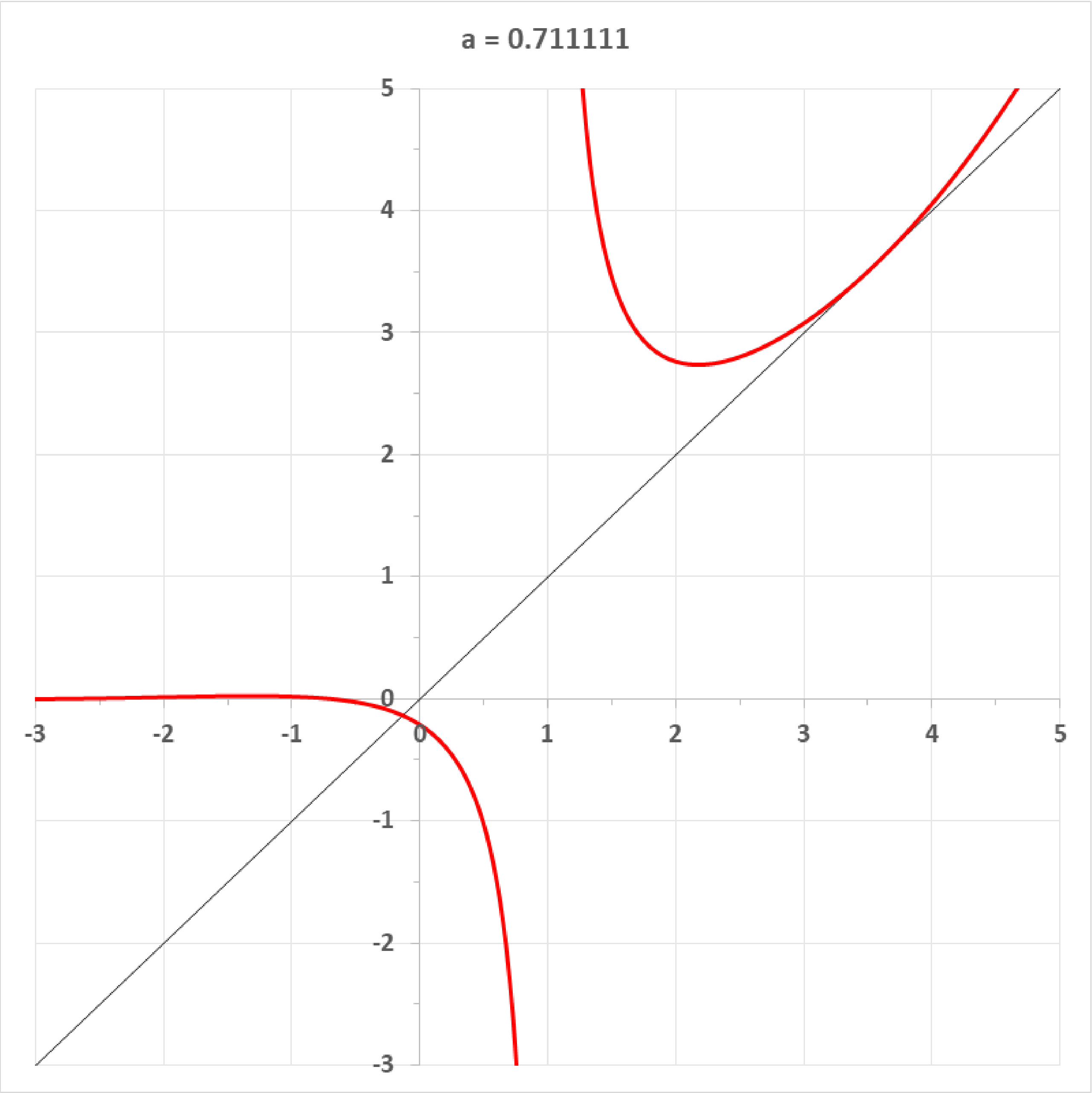
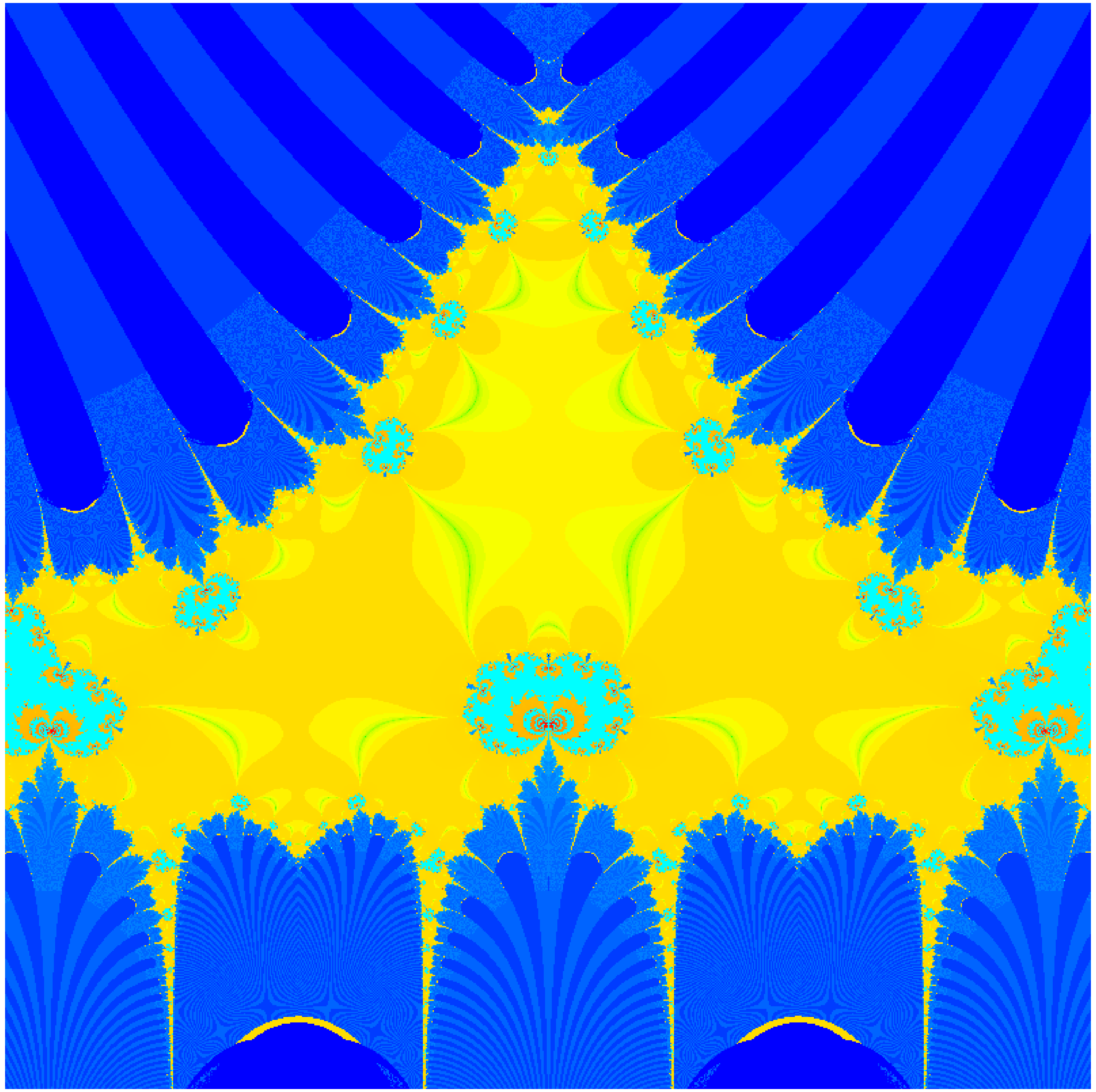
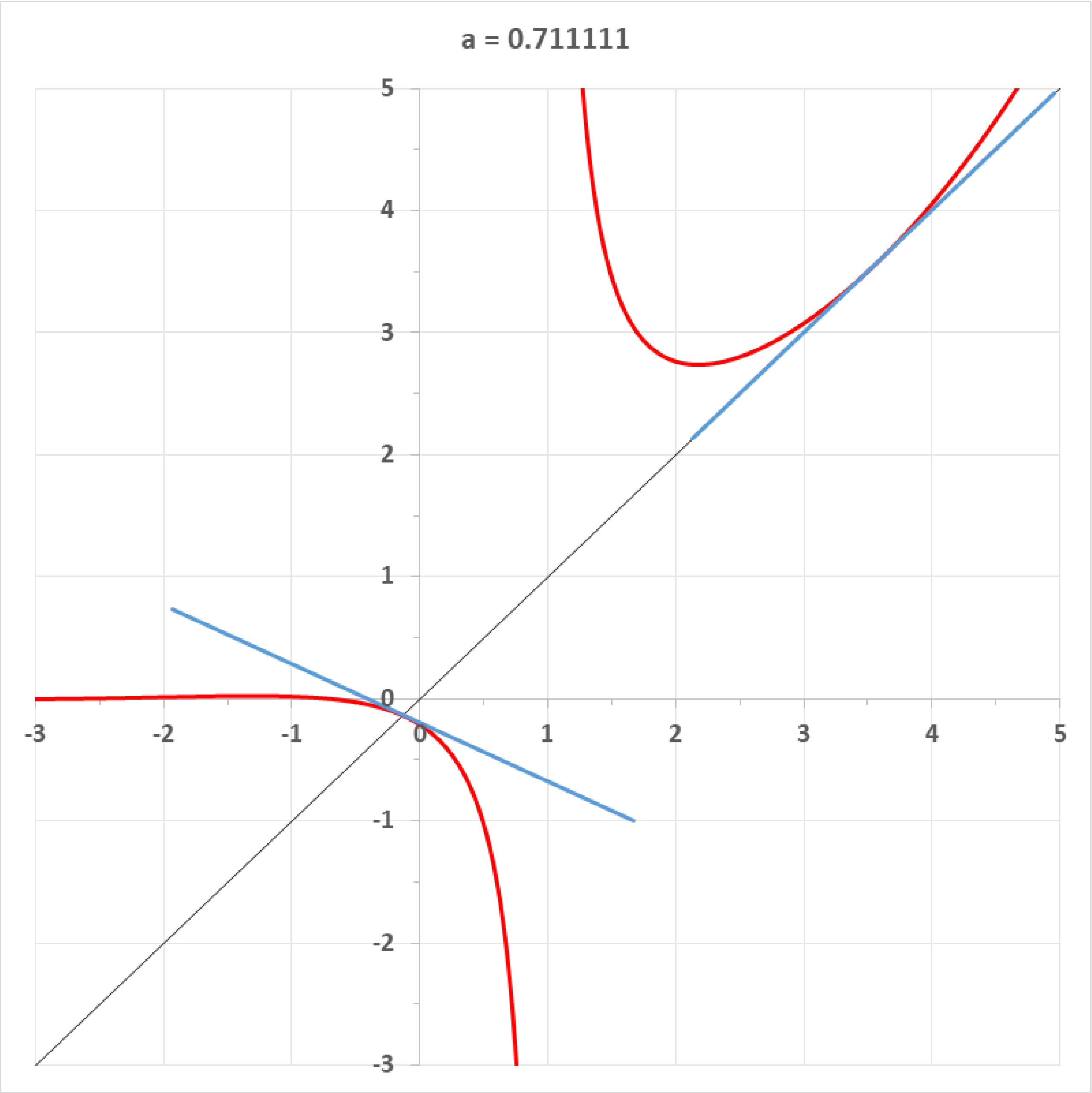
Hurwitz zeta functions

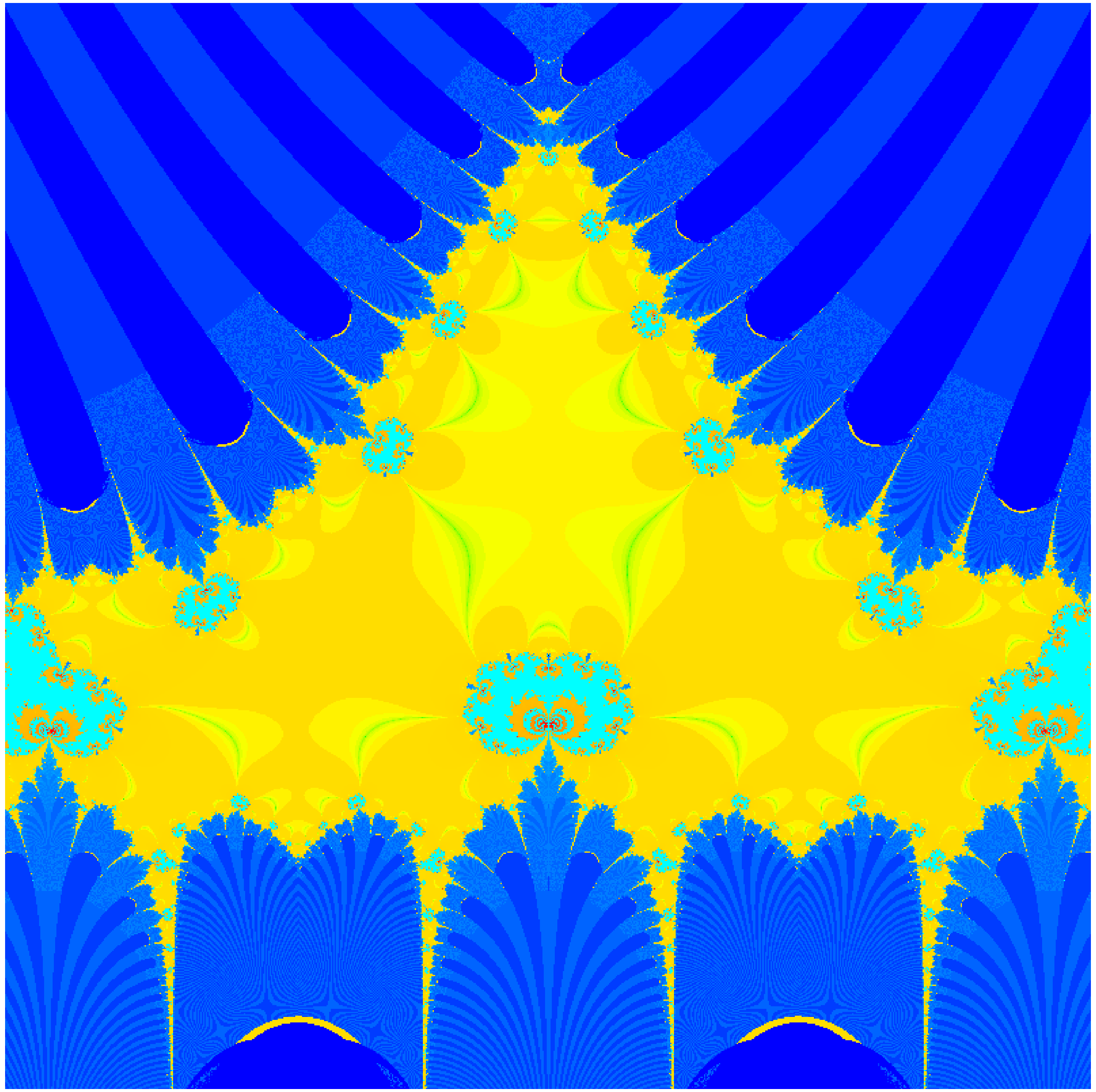




$-23 \leq \Re(s) \leq 31, -20 \leq \Im(s) \leq 20, 20ppu$

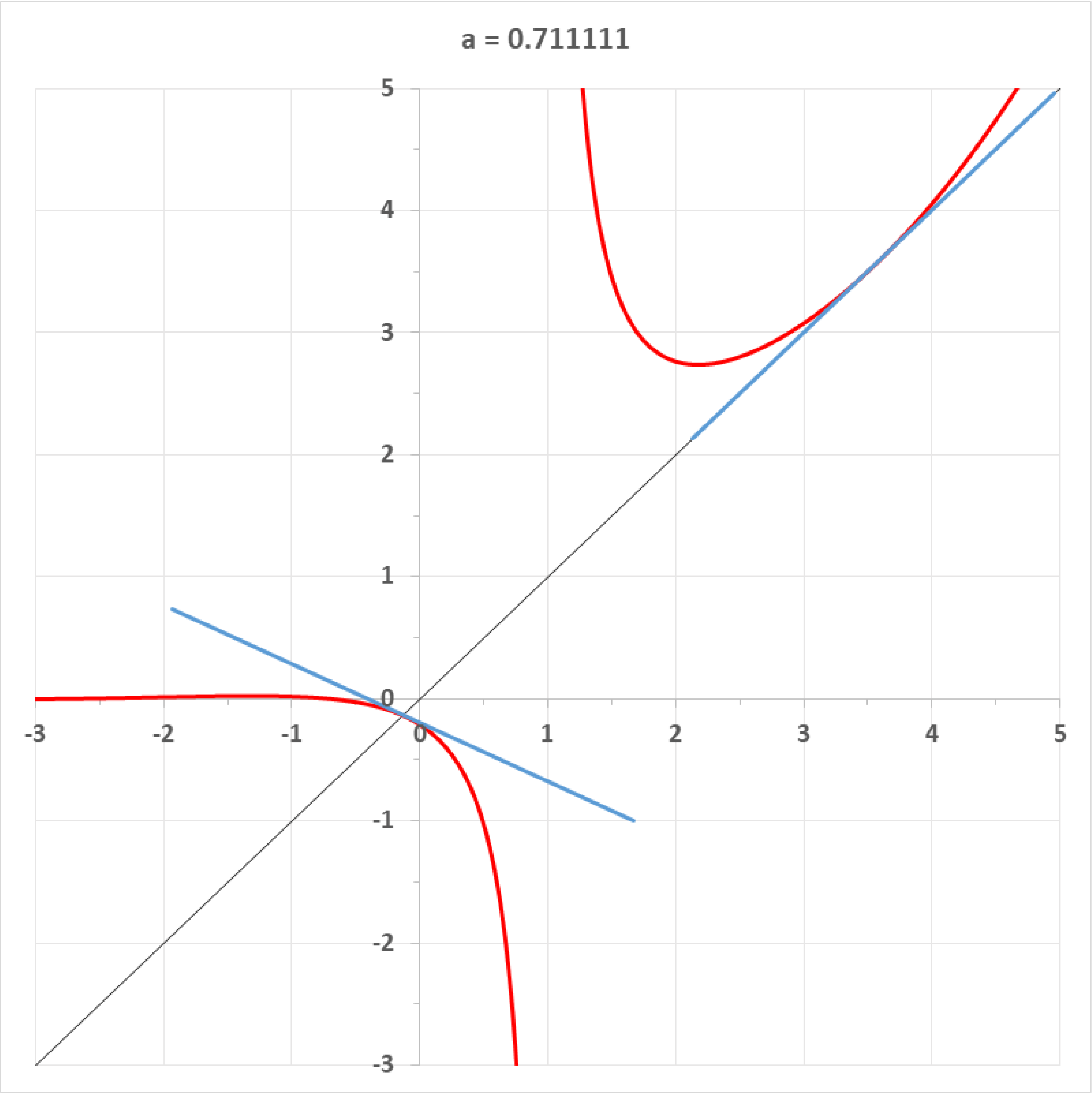
Hurwitz zeta functions

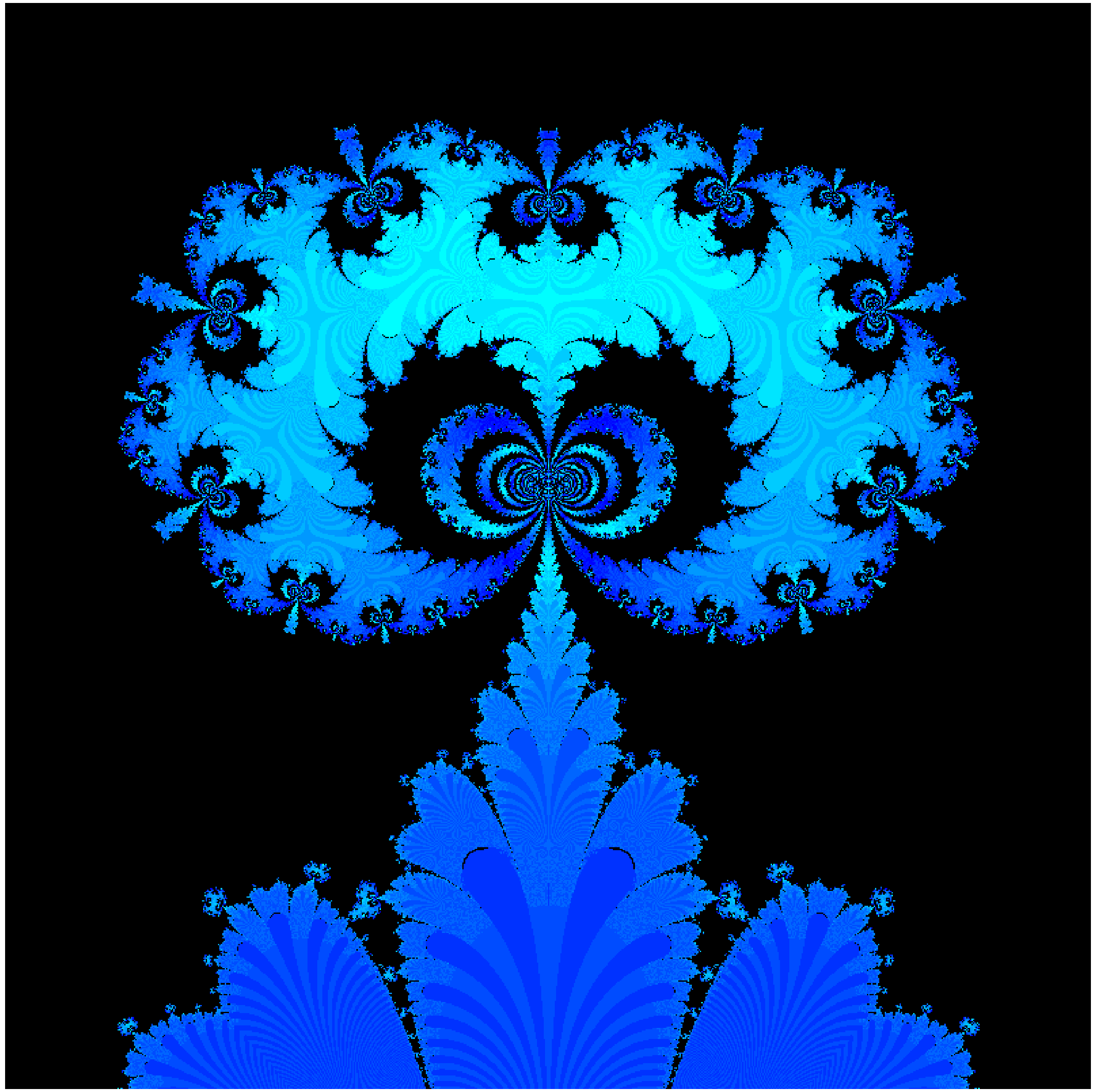




$-23 \leq \Re(s) \leq 31, -20 \leq \Im(s) \leq 20, 20ppu$

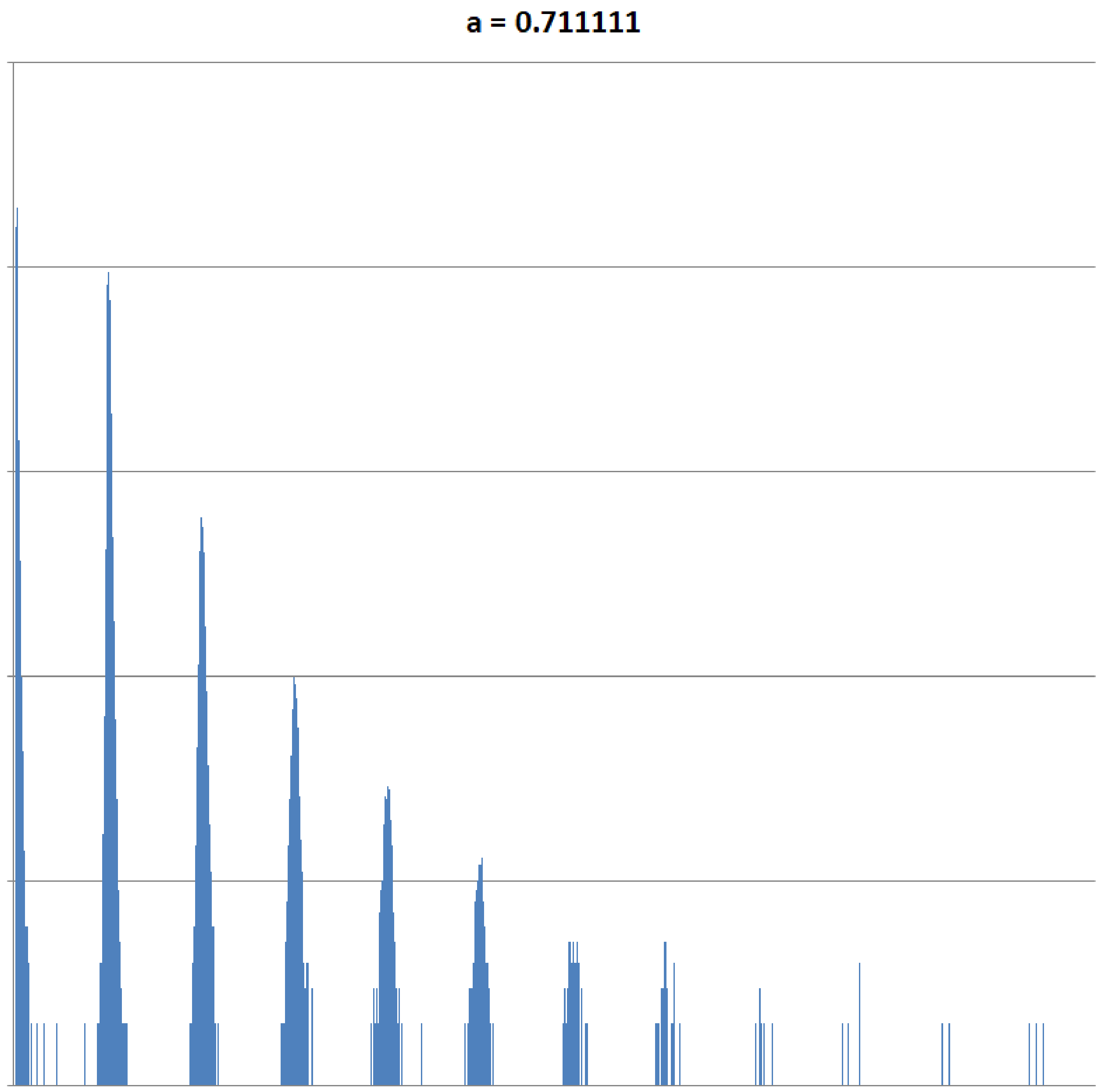
Hurwitz zeta functions

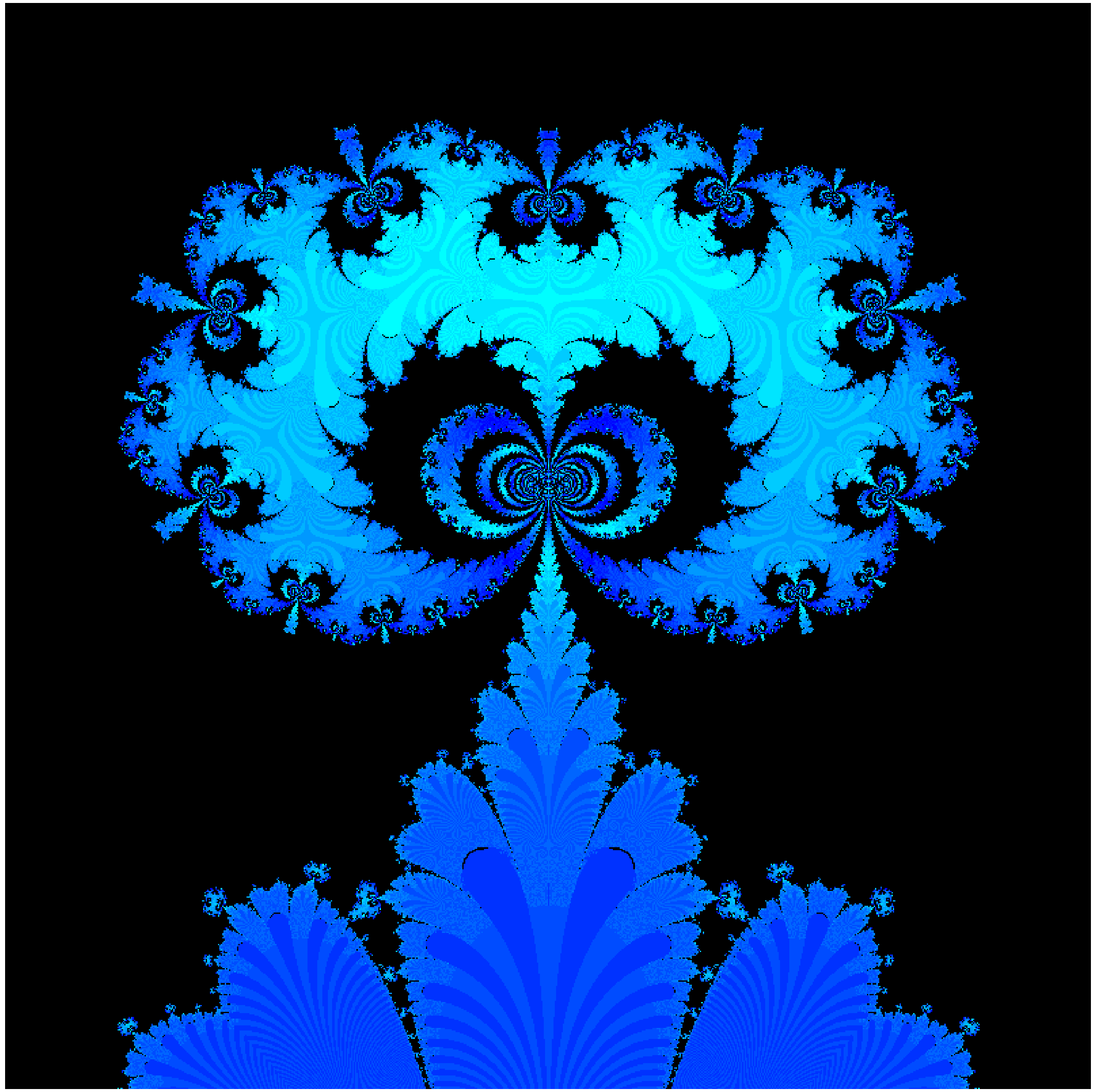




$-1 \leq \Re(s) \leq 9, -5 \leq \Im(s) \leq 5, 80ppu$

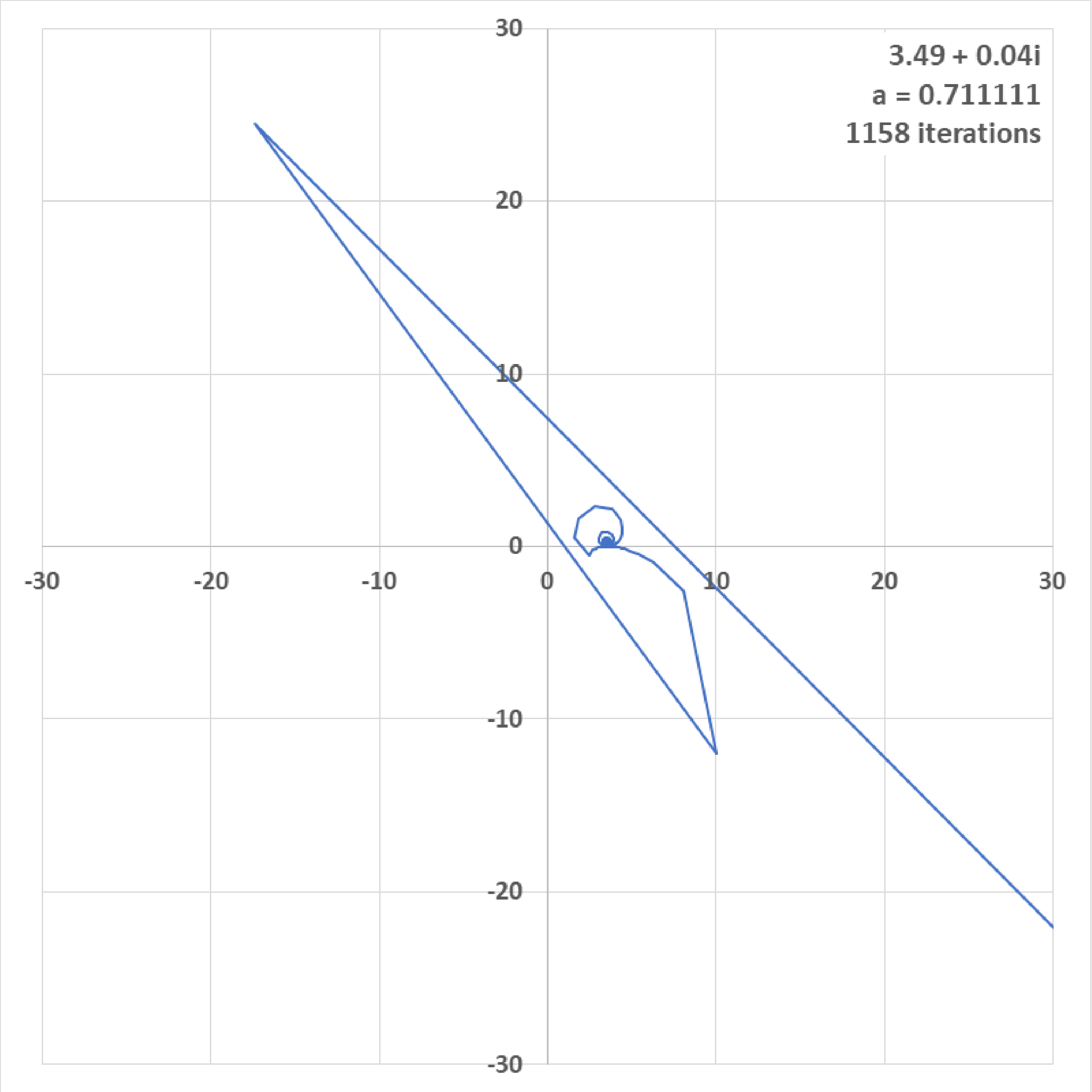
Hurwitz zeta functions

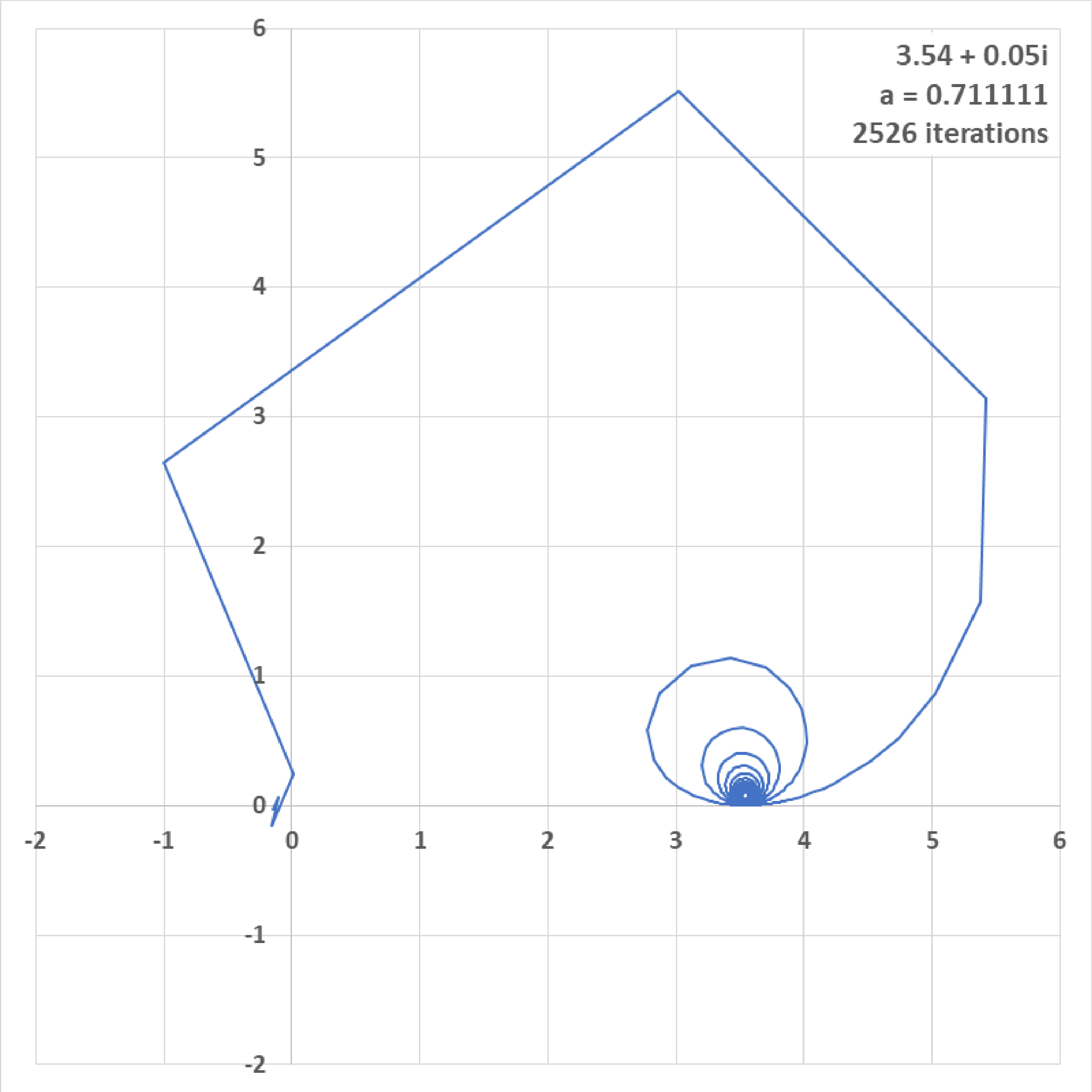




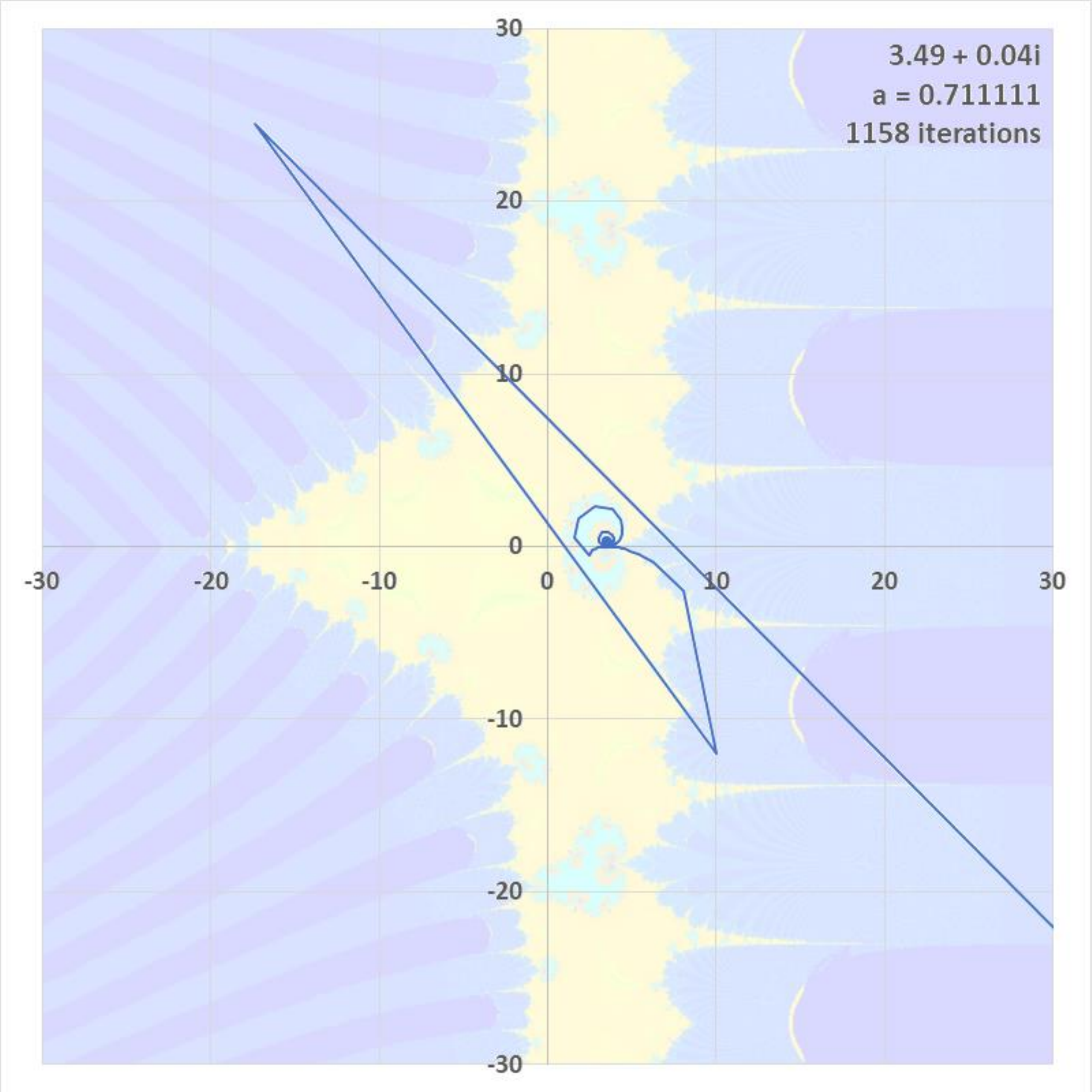
$-1 \leq \Re(s) \leq 9, -5 \leq \Im(s) \leq 5, 80ppu$

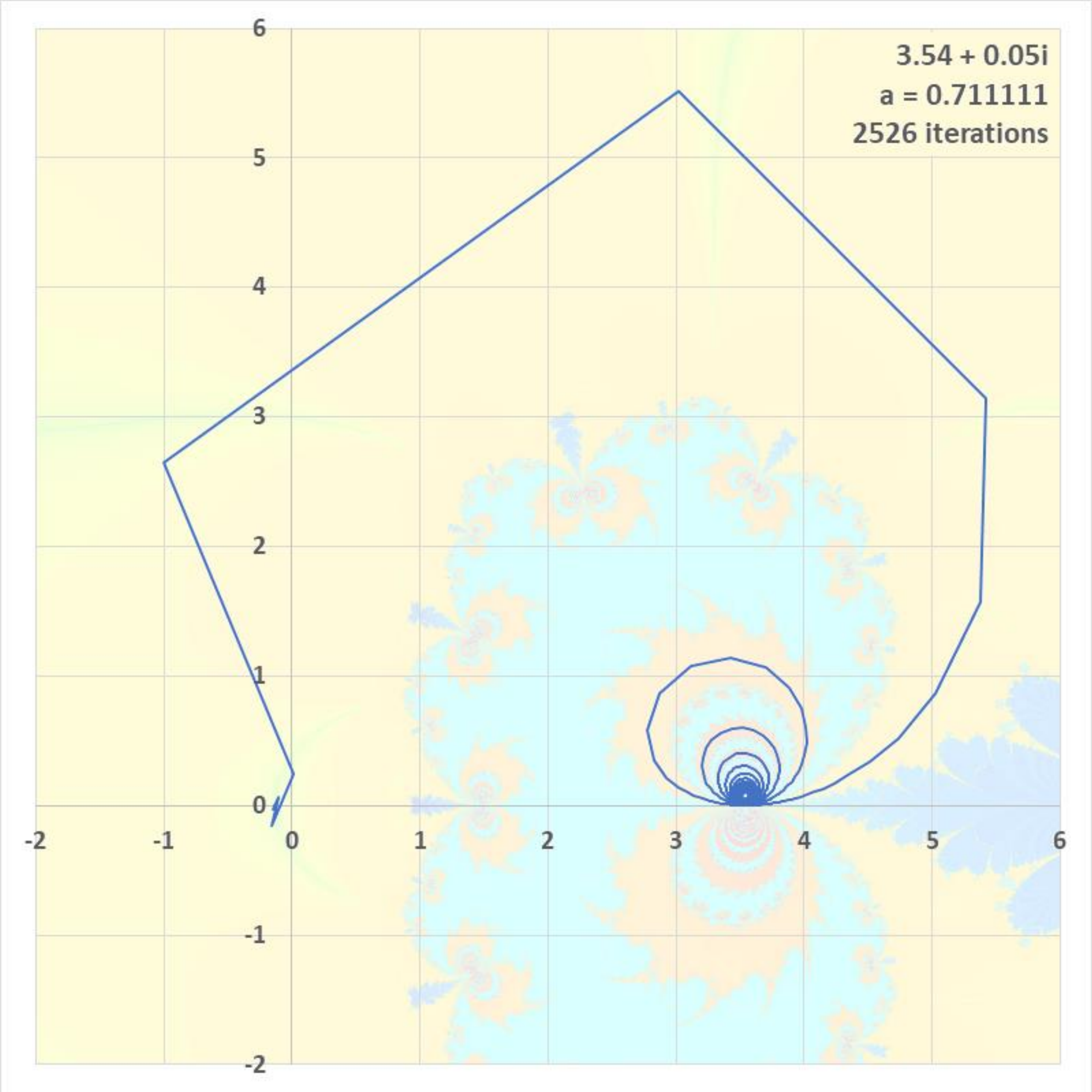
Hurwitz zeta functions





Hurwitz zeta functions





Dirichlet L functions

The Dirchlet L series of a Dirchlet character χ of modulus k and $s \in \mathbb{C}$ is defined as:

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad \Re(s) > 1 \tag{1}$$

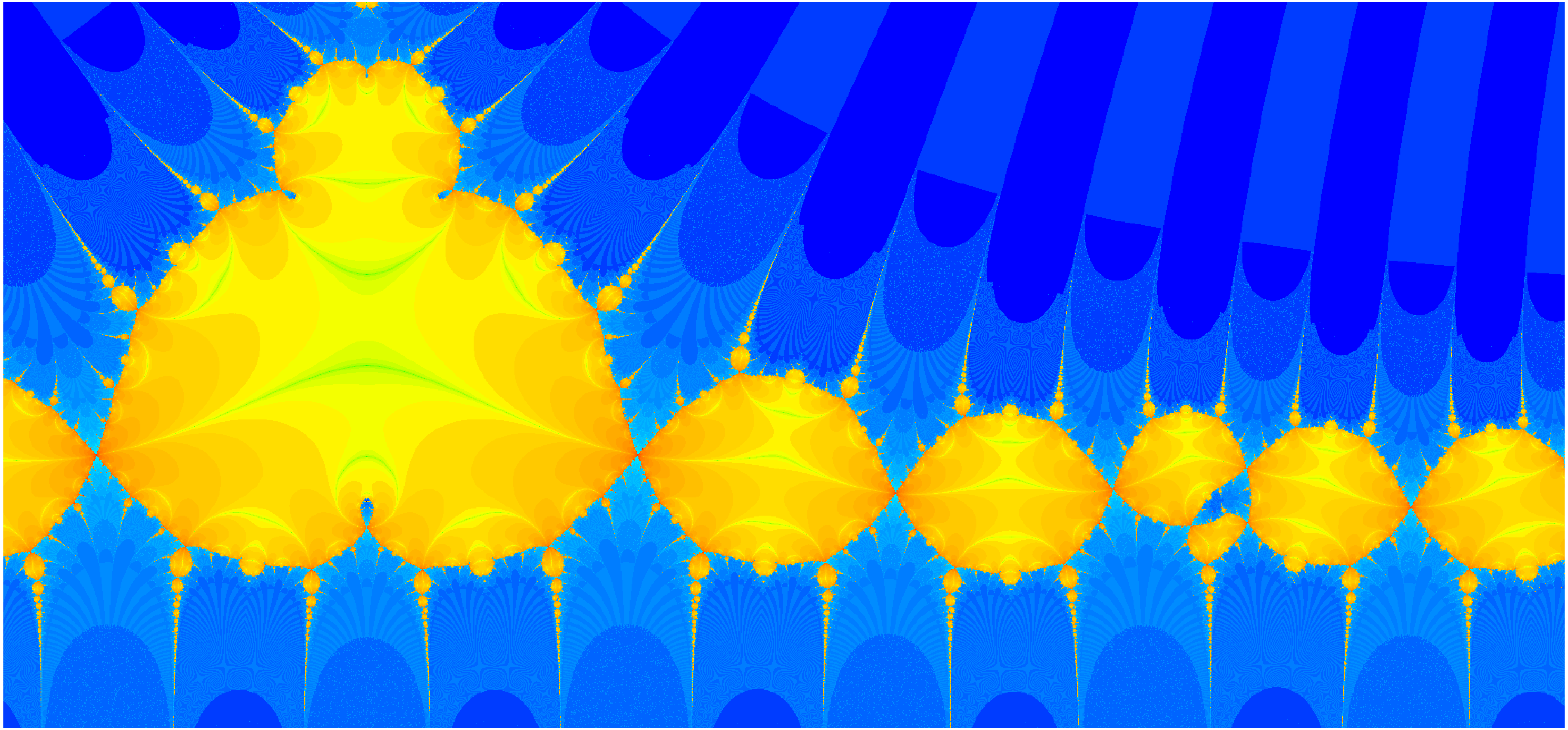
The function can be analytically continued to the whole complex plane, where it is known as a Dirchlet L function. Where χ is a principal character (i.e. χ assumes the value 1 for arguments coprime to its modulus k, and 0 otherwise), then the function has a simple pole at $s = 1$.

For a given Dirichlet character χ modulo k, the corresponding Dirichlet L function can be expressed as a linear combination of Hurwitz zeta functions $\zeta(s,a)$ with $a \in \mathbb{Q}$:

$$L(s,\chi) = \frac{1}{k^s} \sum_{m=1}^{k} \chi(m)\, \zeta\left(s, \frac{m}{k}\right) \qquad k, m \in \mathbb{Z} \tag{2}$$

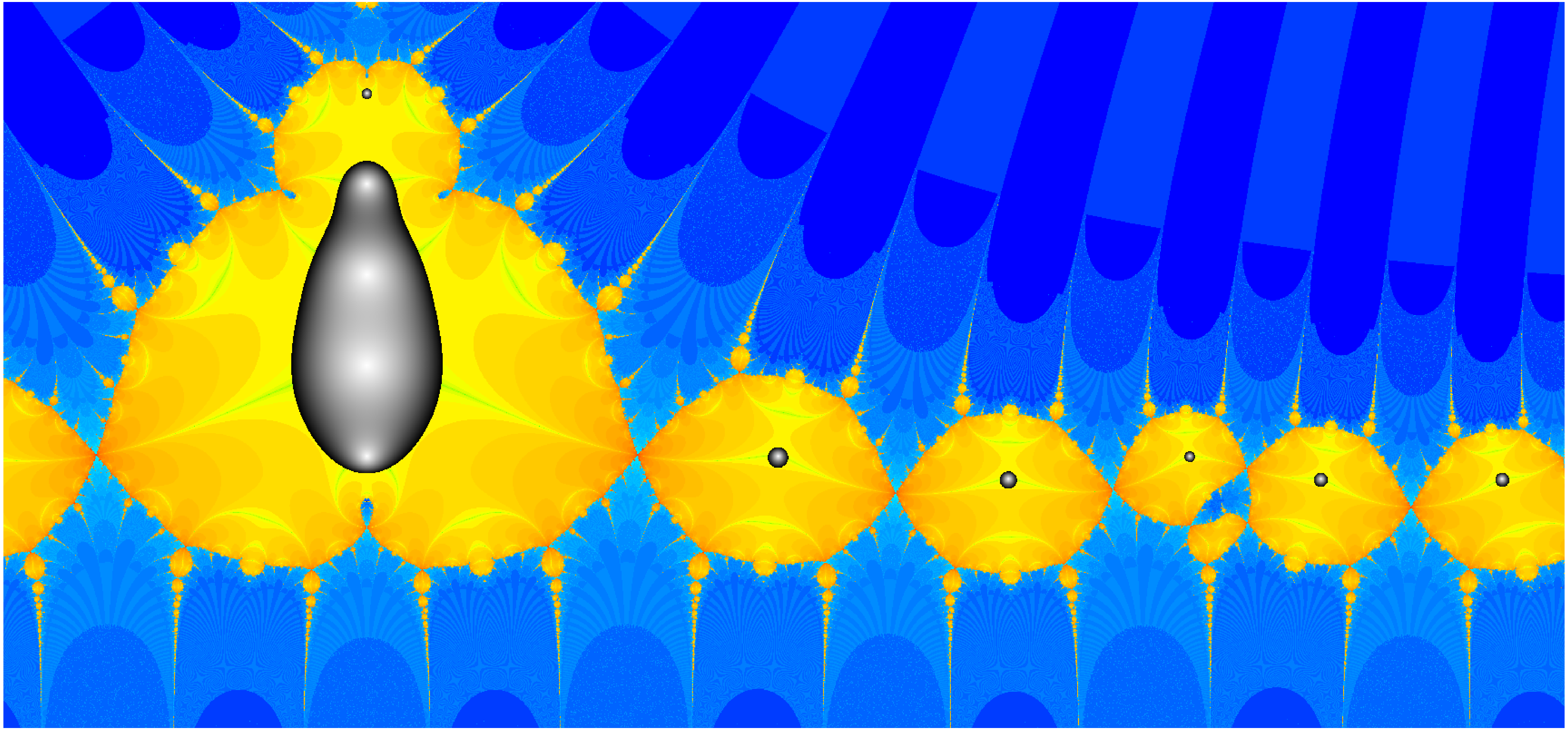
Approximations of Dirichlet L functions by `zeta_machine` relate to principal characters only. Precision is limited to about 10 significant figures.

Dirichlet L function: χ_1 mod 2



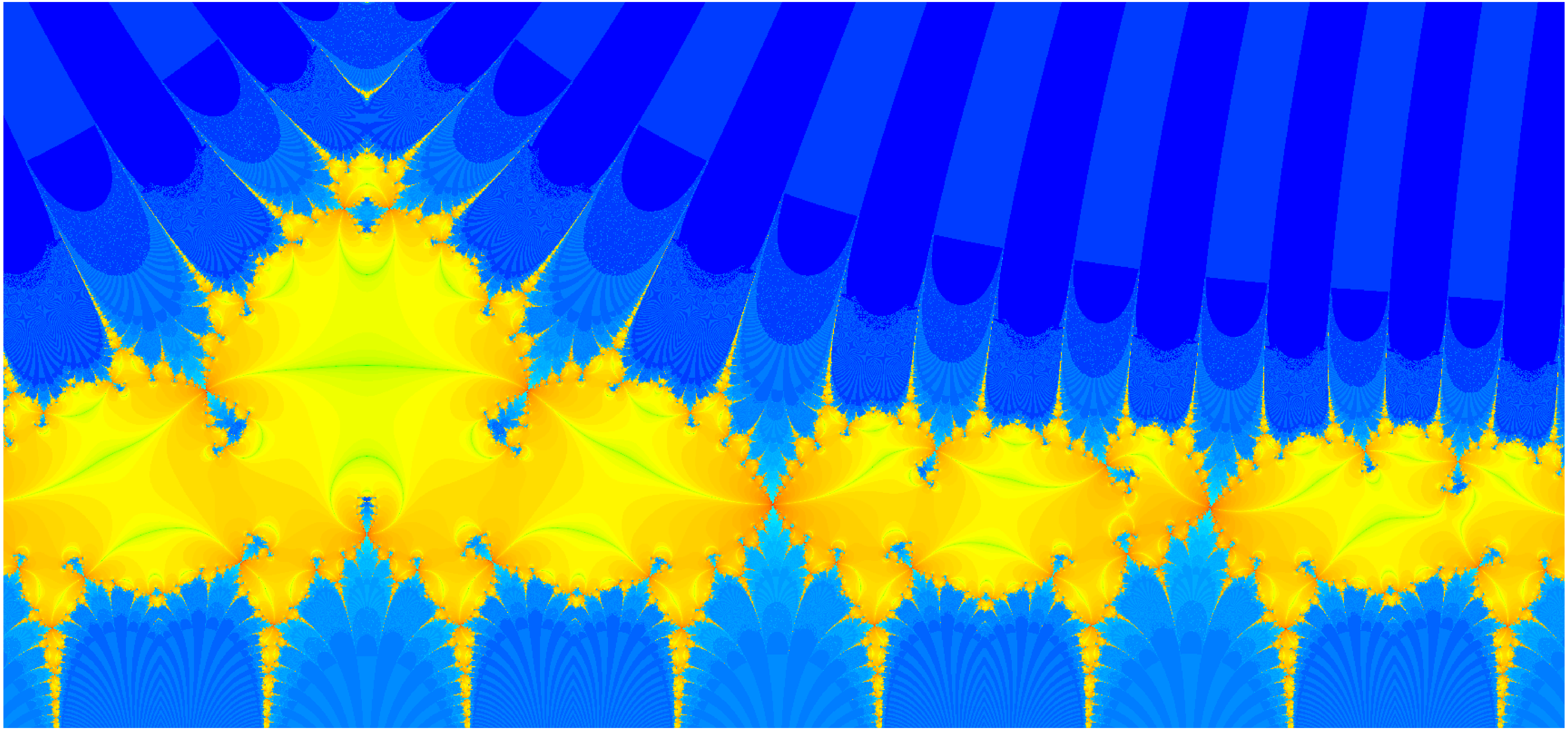
$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dirichlet L function: χ_1 mod 2



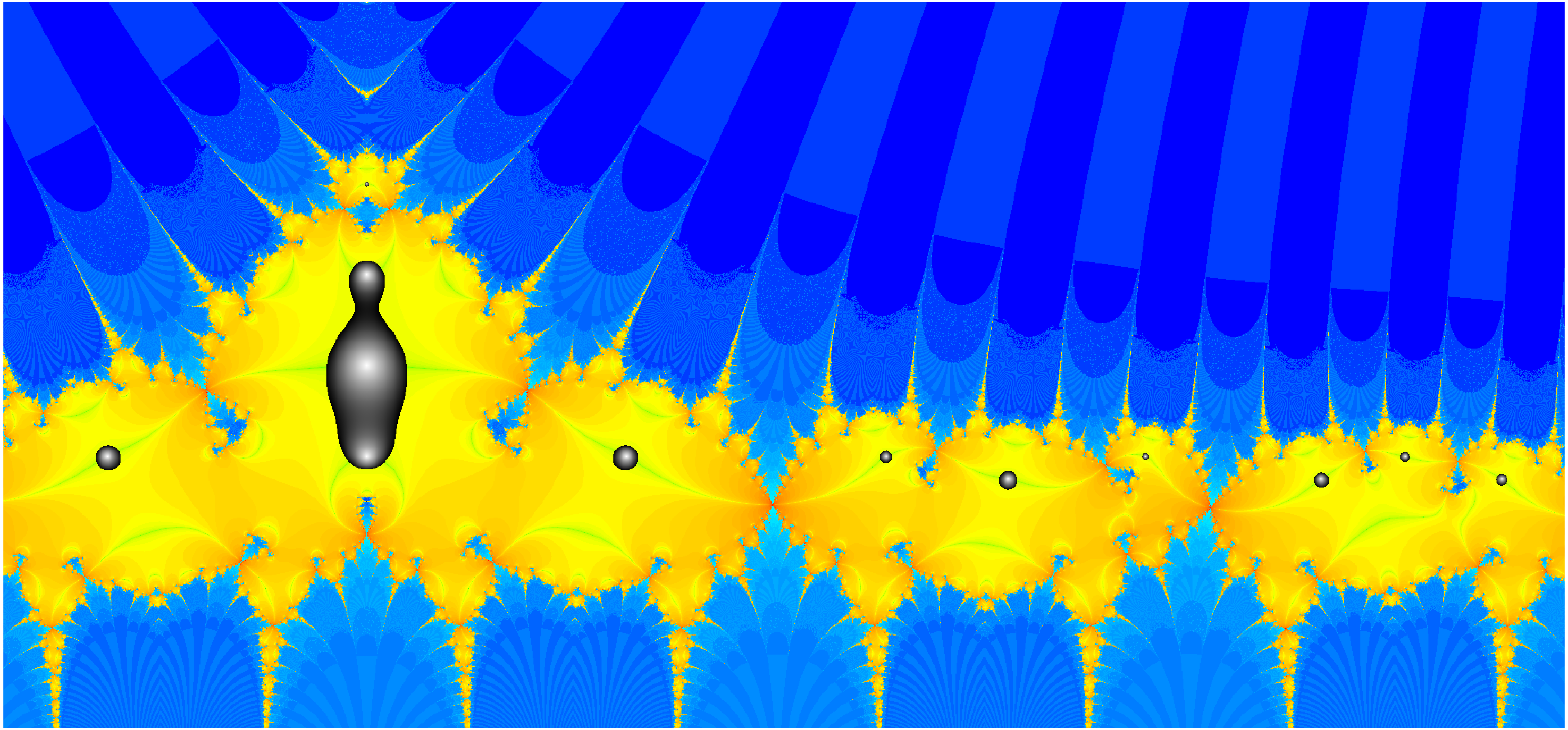
$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dirichlet L function: χ_1 mod 3



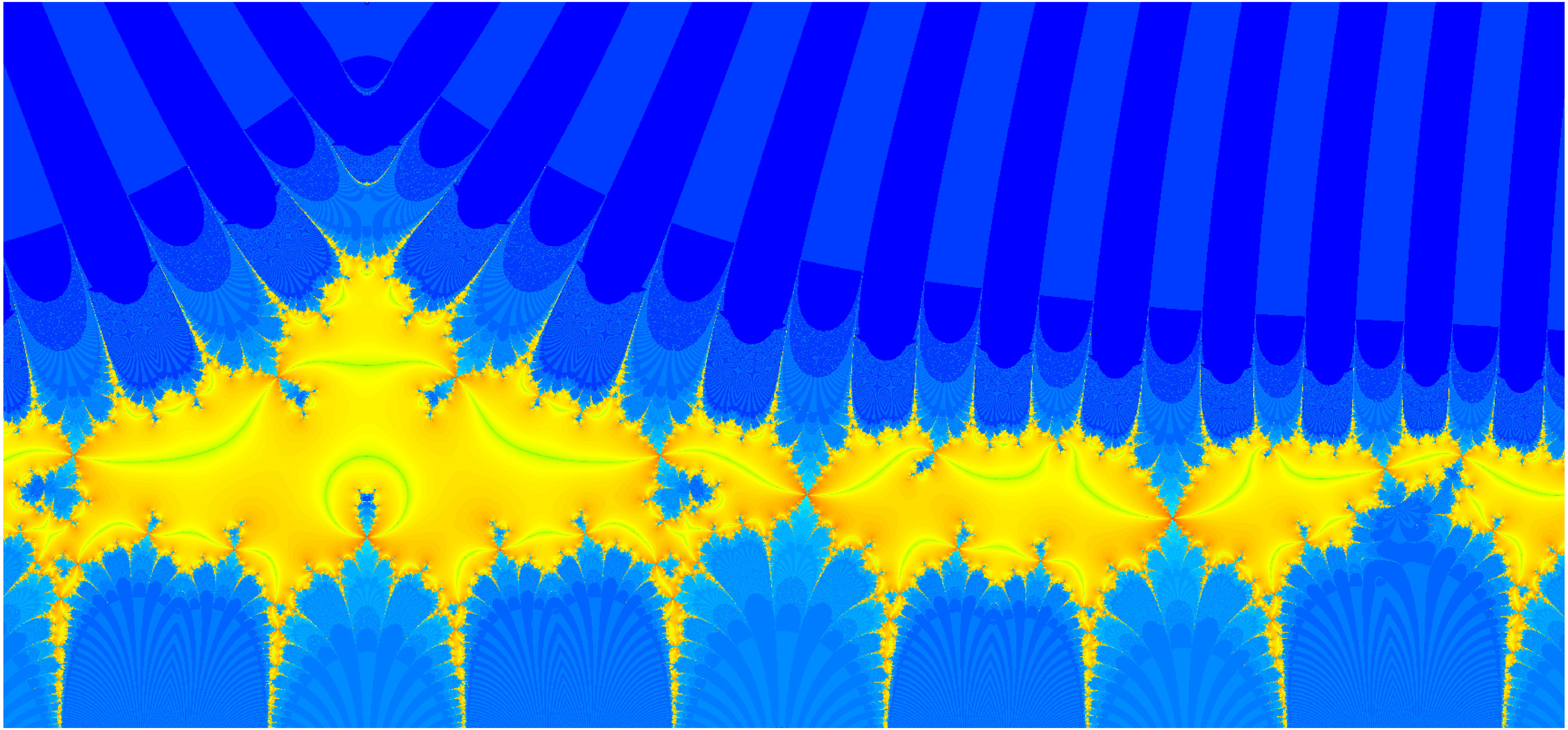
$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dirichlet L function: χ_1 mod 3



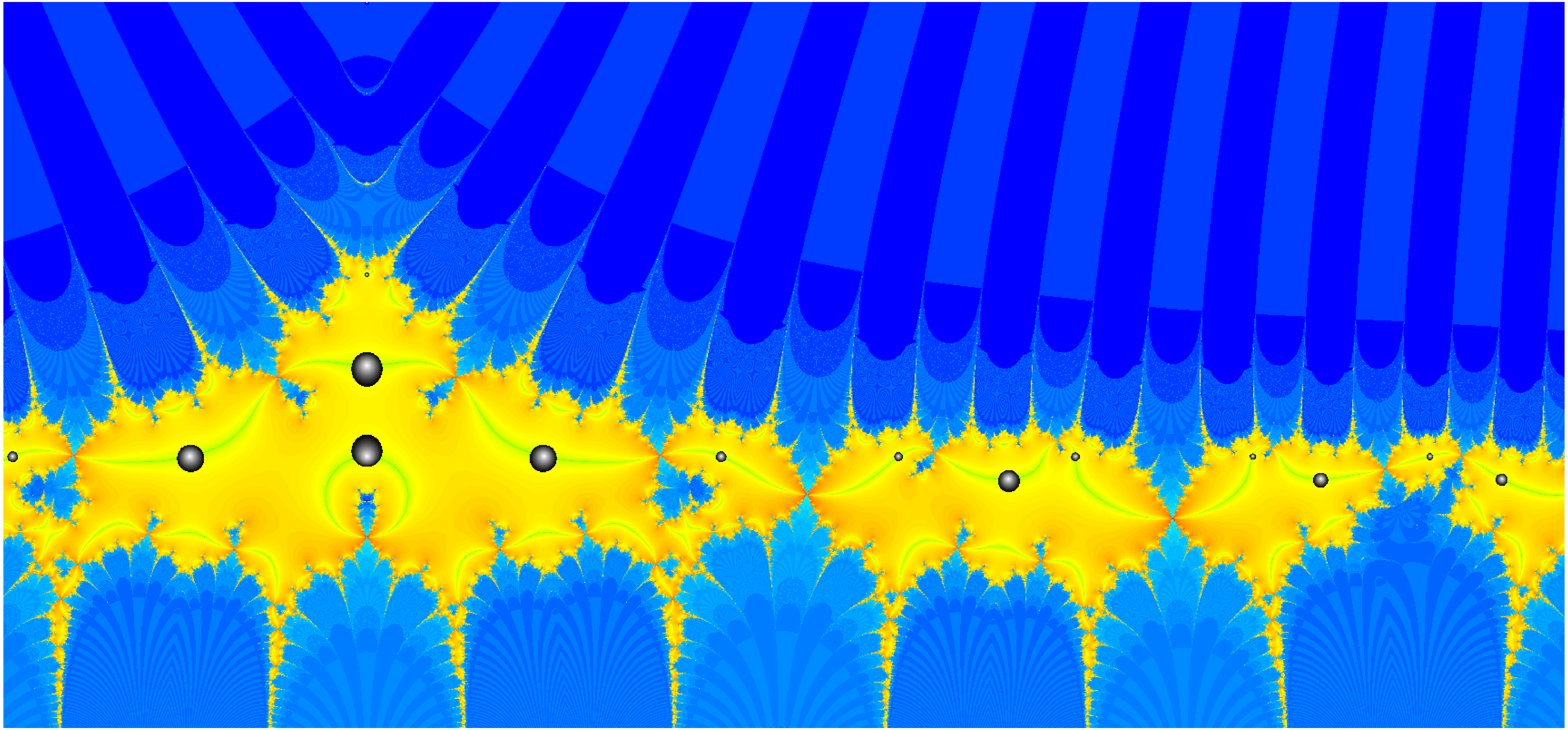
$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dirichlet L function: χ_1 mod 5



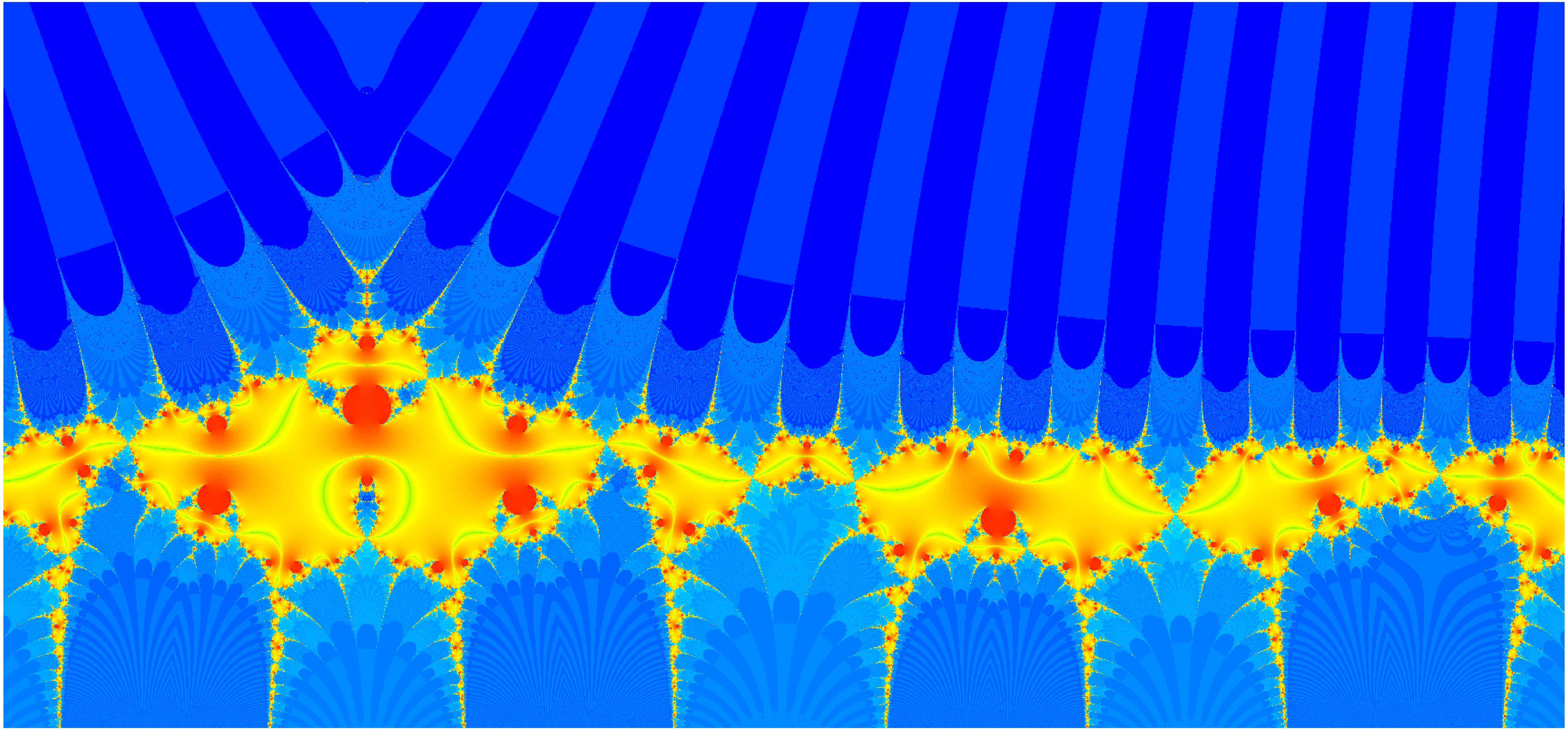
$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dirichlet L function: χ_1 mod 5



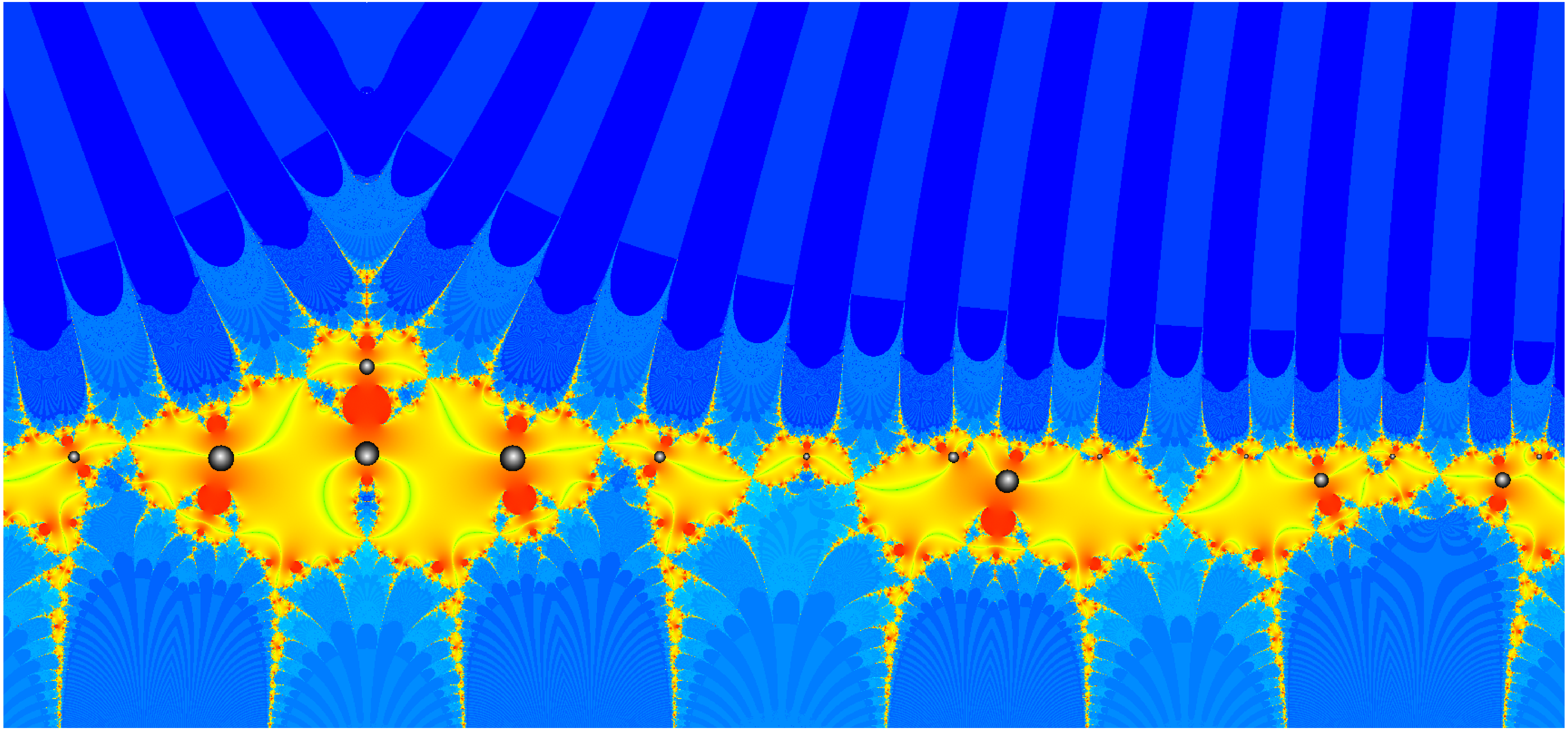
$-10 \le \Re(s) \le 6, -0.8 \le \Im(s) \le 26.5, 50ppu$

Dirichlet L function: χ_1 mod 7



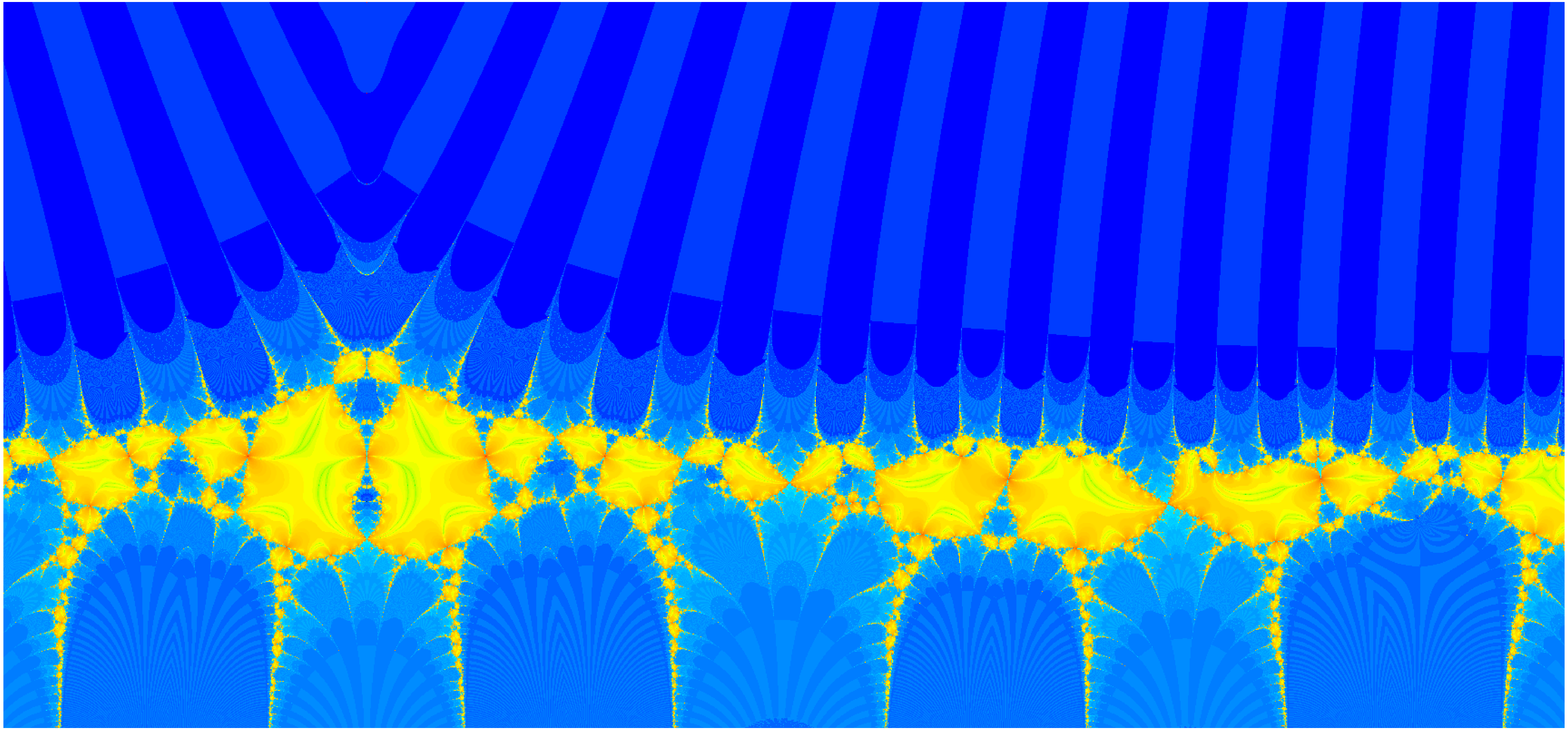
$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dirichlet L function: χ_1 mod 7



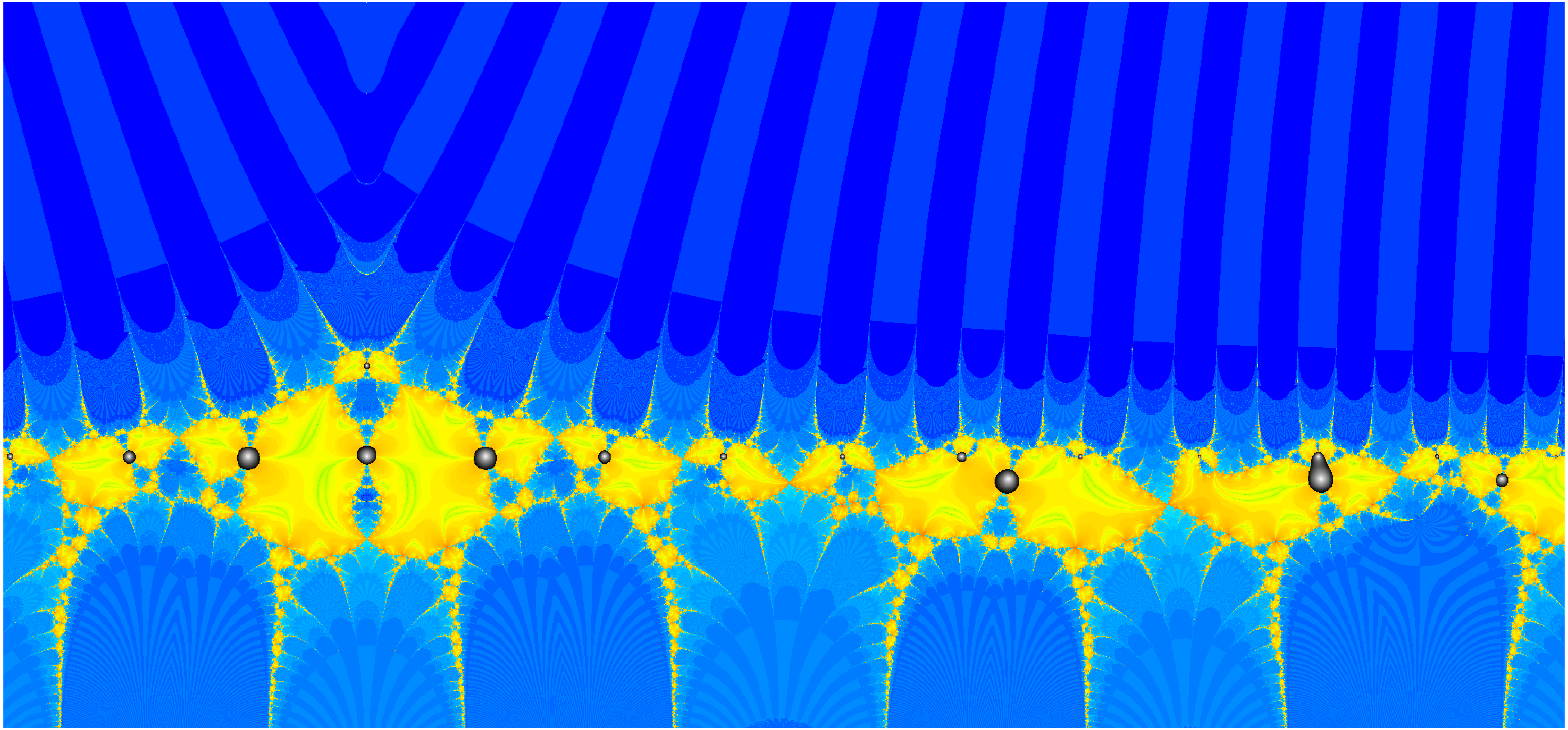
$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dirichlet L function: χ_1 mod 11



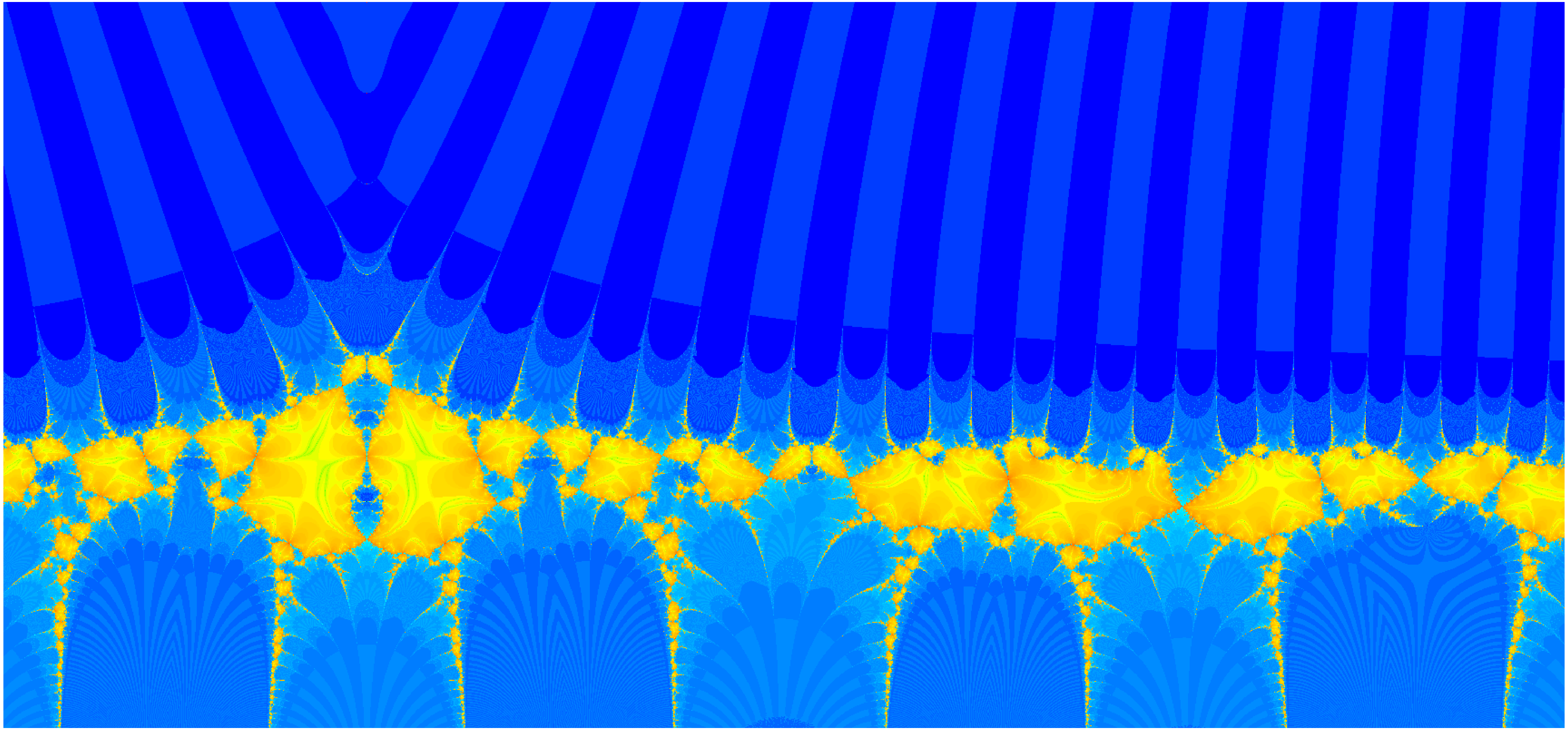
$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dirichlet L function: χ_1 mod 11



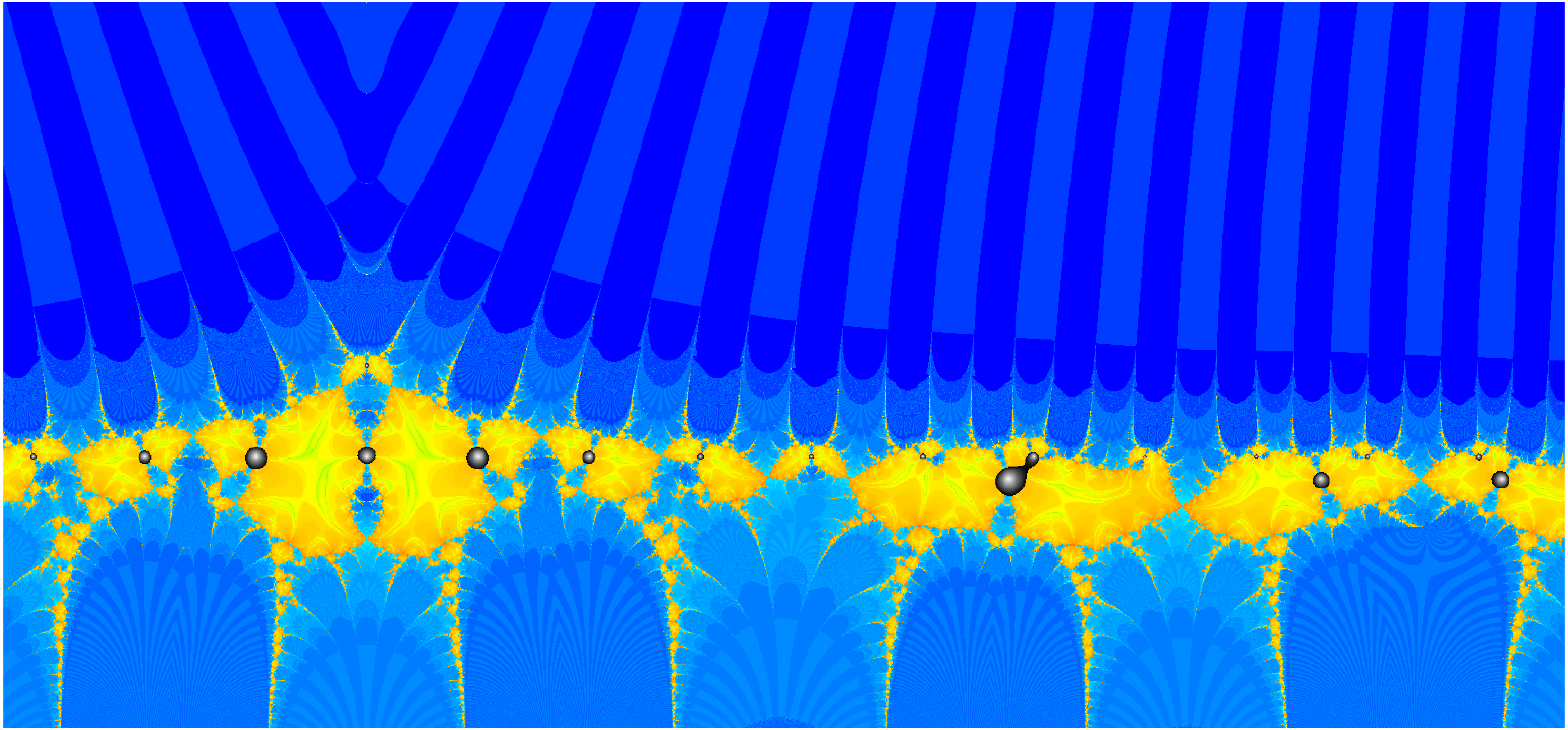
$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dirichlet L function: χ_1 mod 13



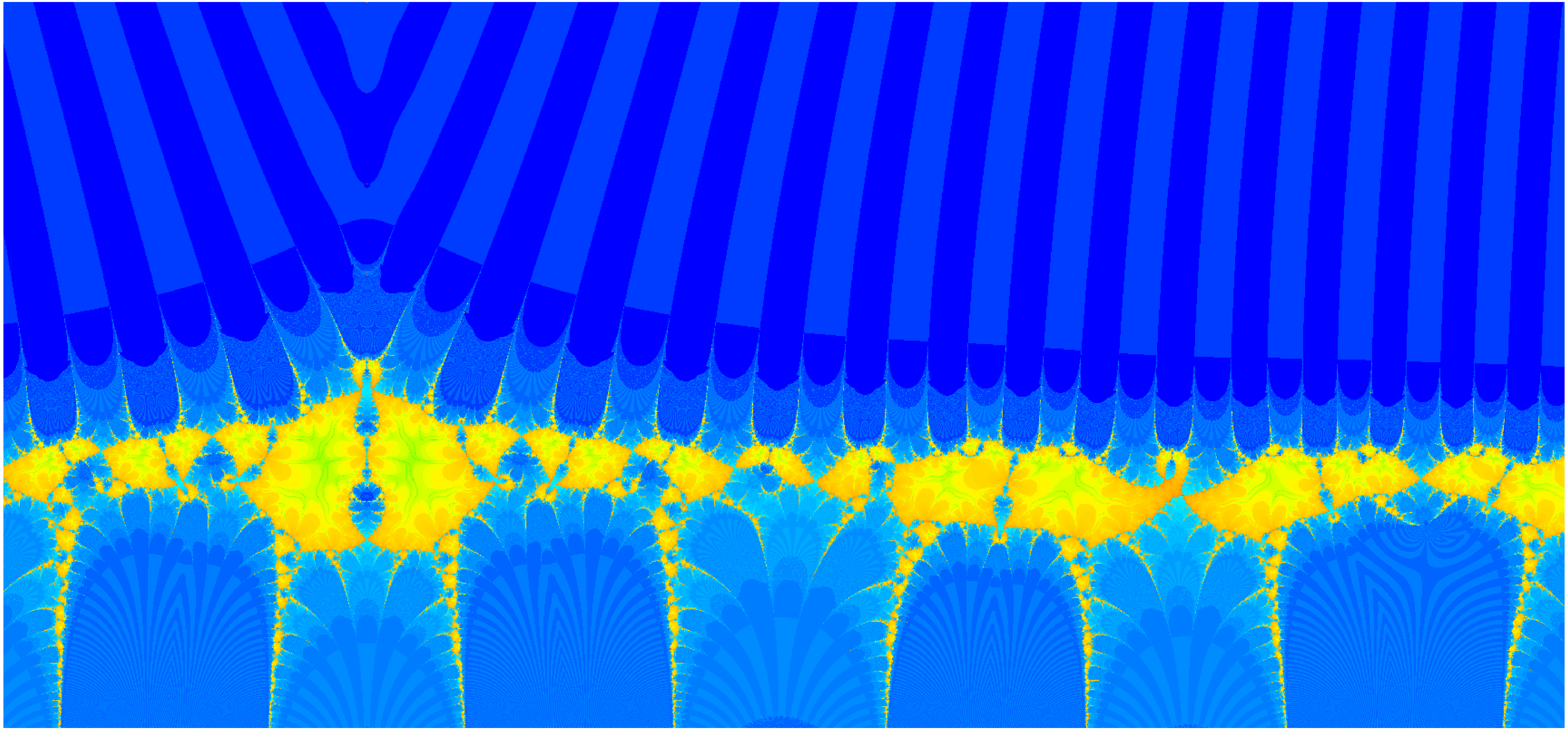
$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dirichlet L function: χ_1 mod 13



$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dirichlet L function: χ_1 mod 17



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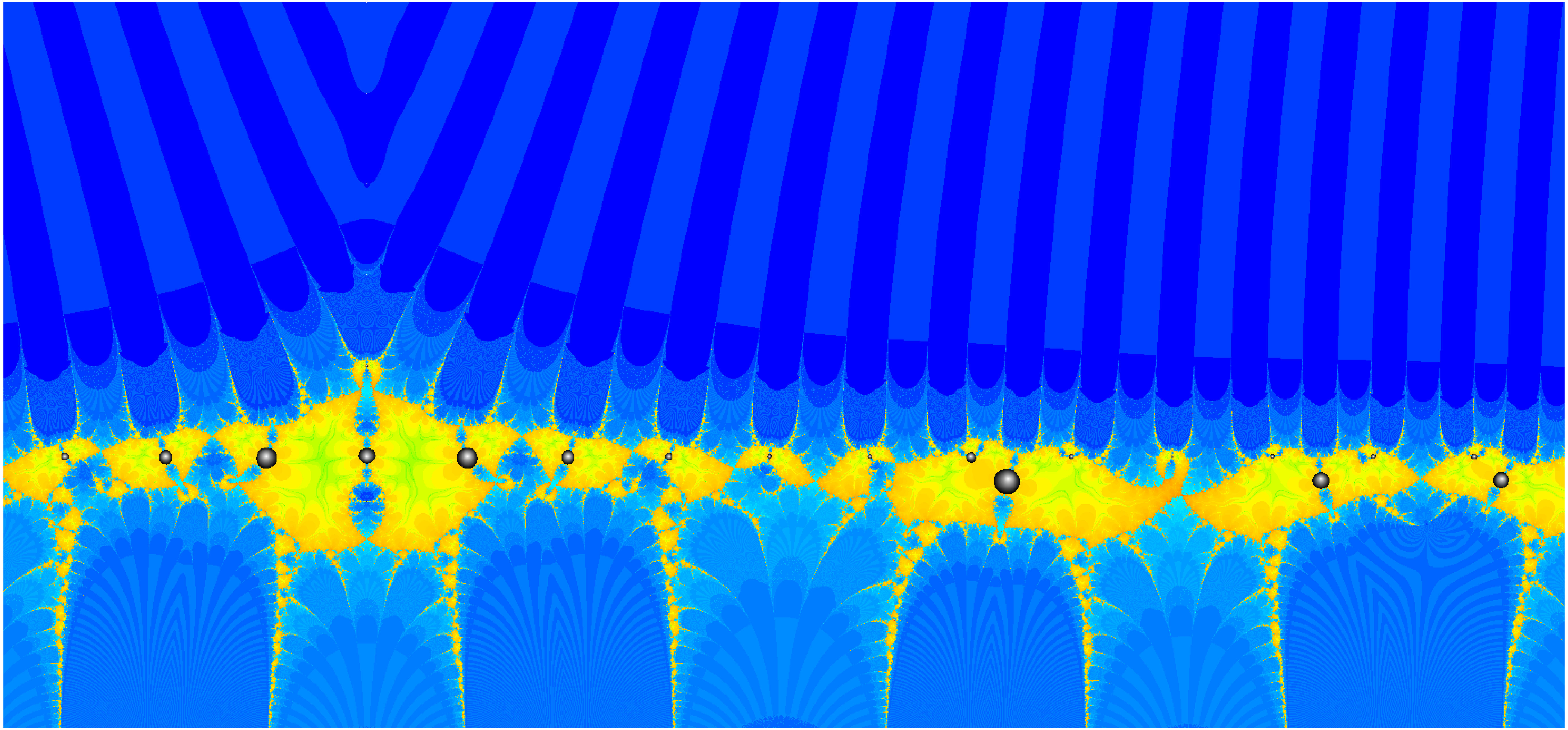
Dirichlet L function: χ_1 mod 17



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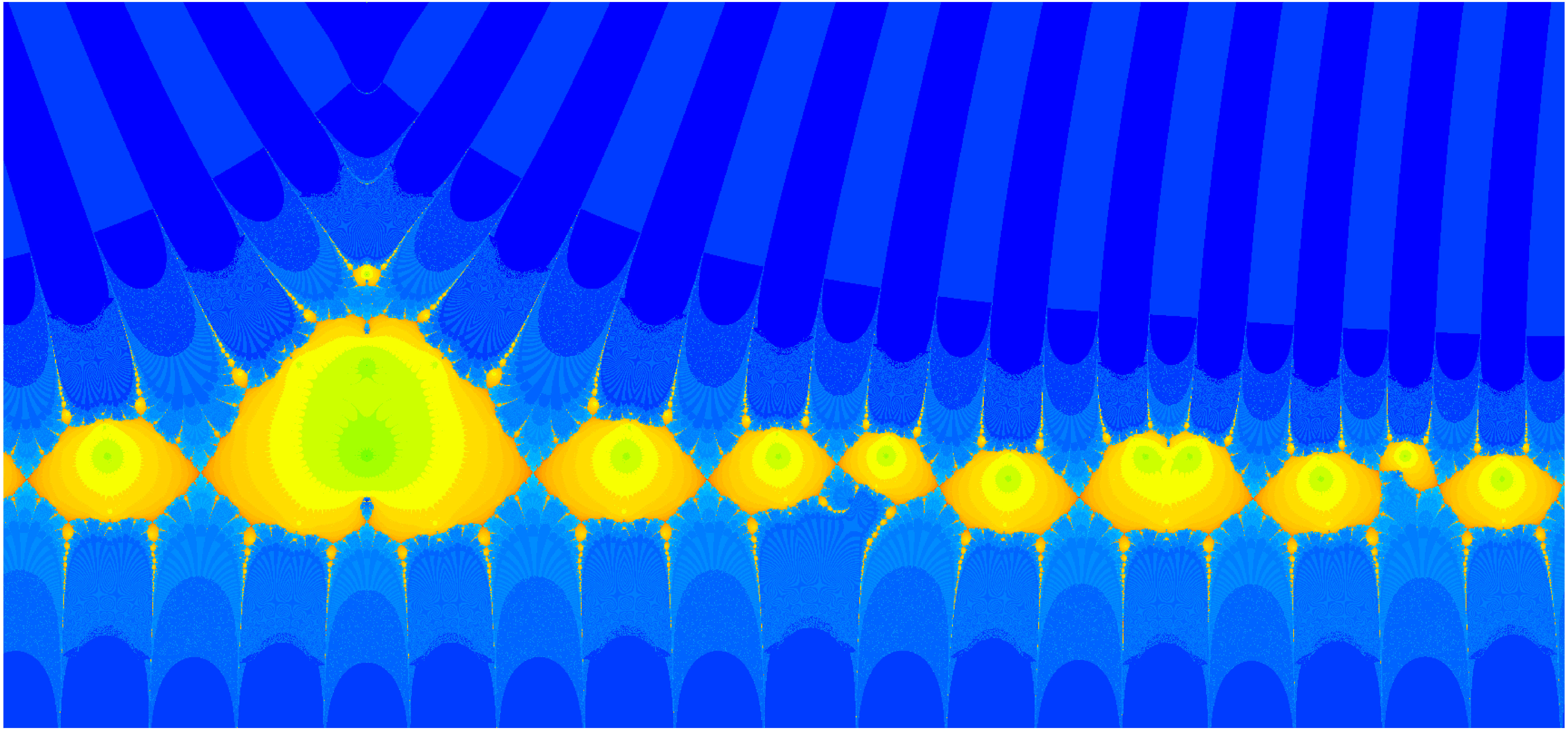
Dirichlet L function: χ_1 mod 6



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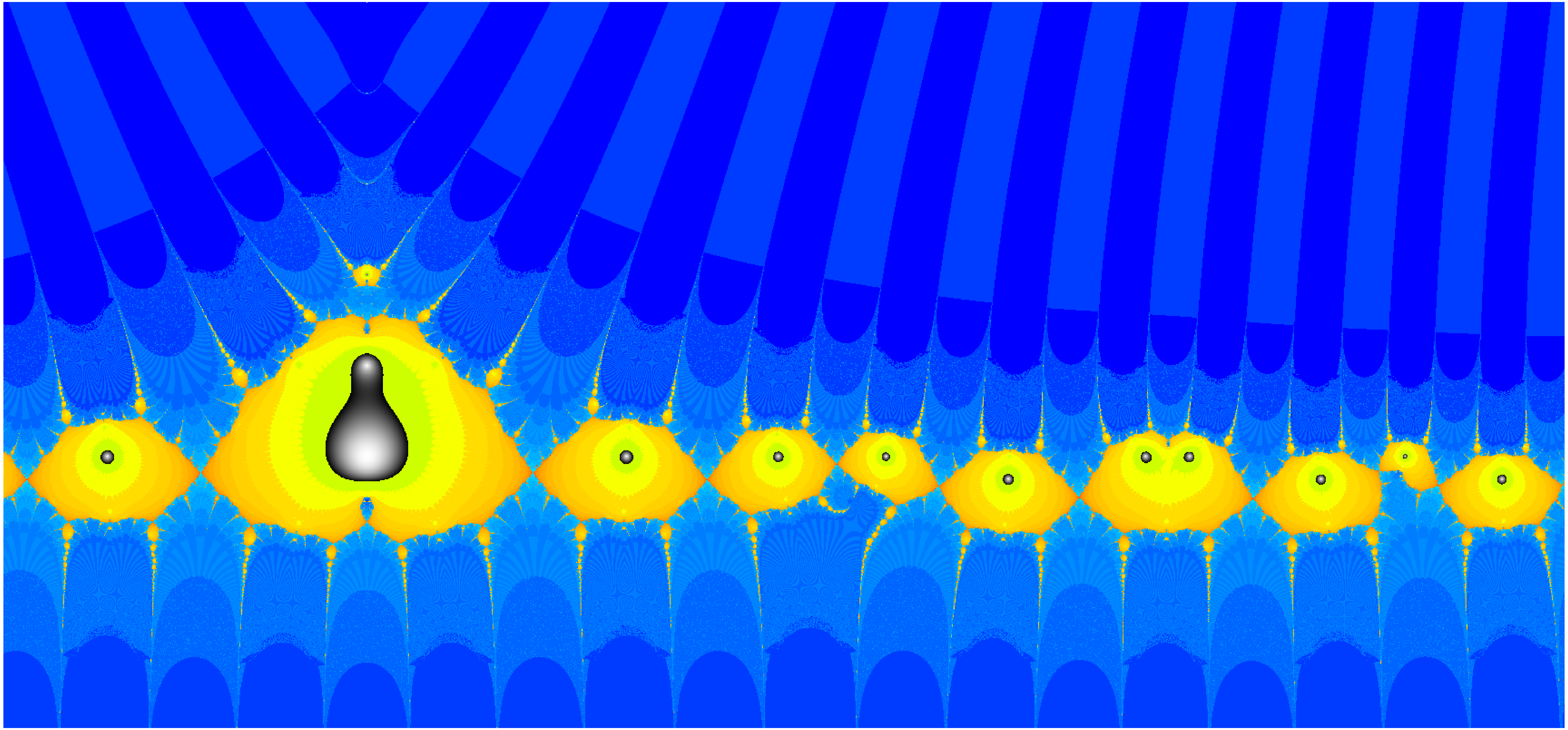
Dirichlet L function: χ_1 mod 6



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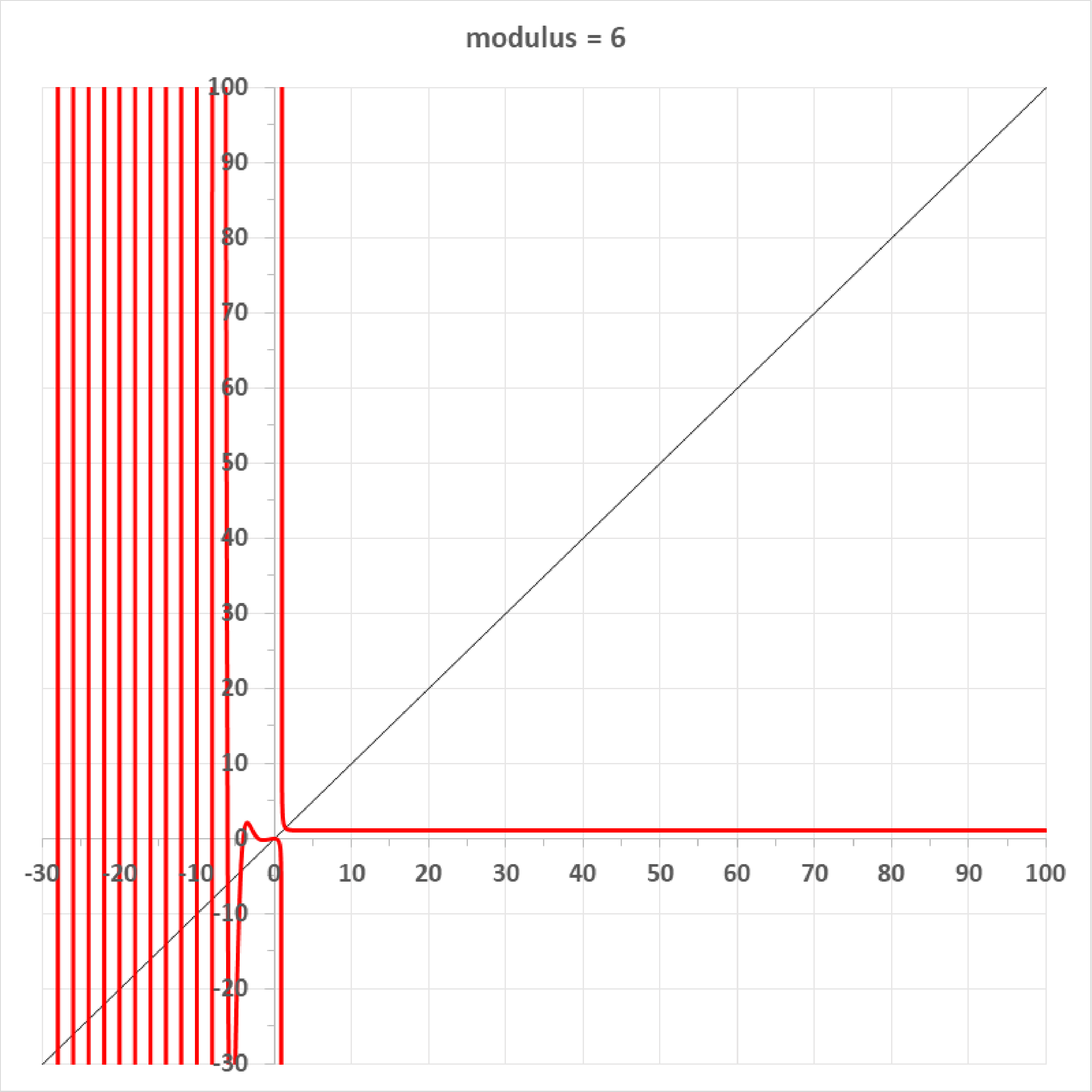
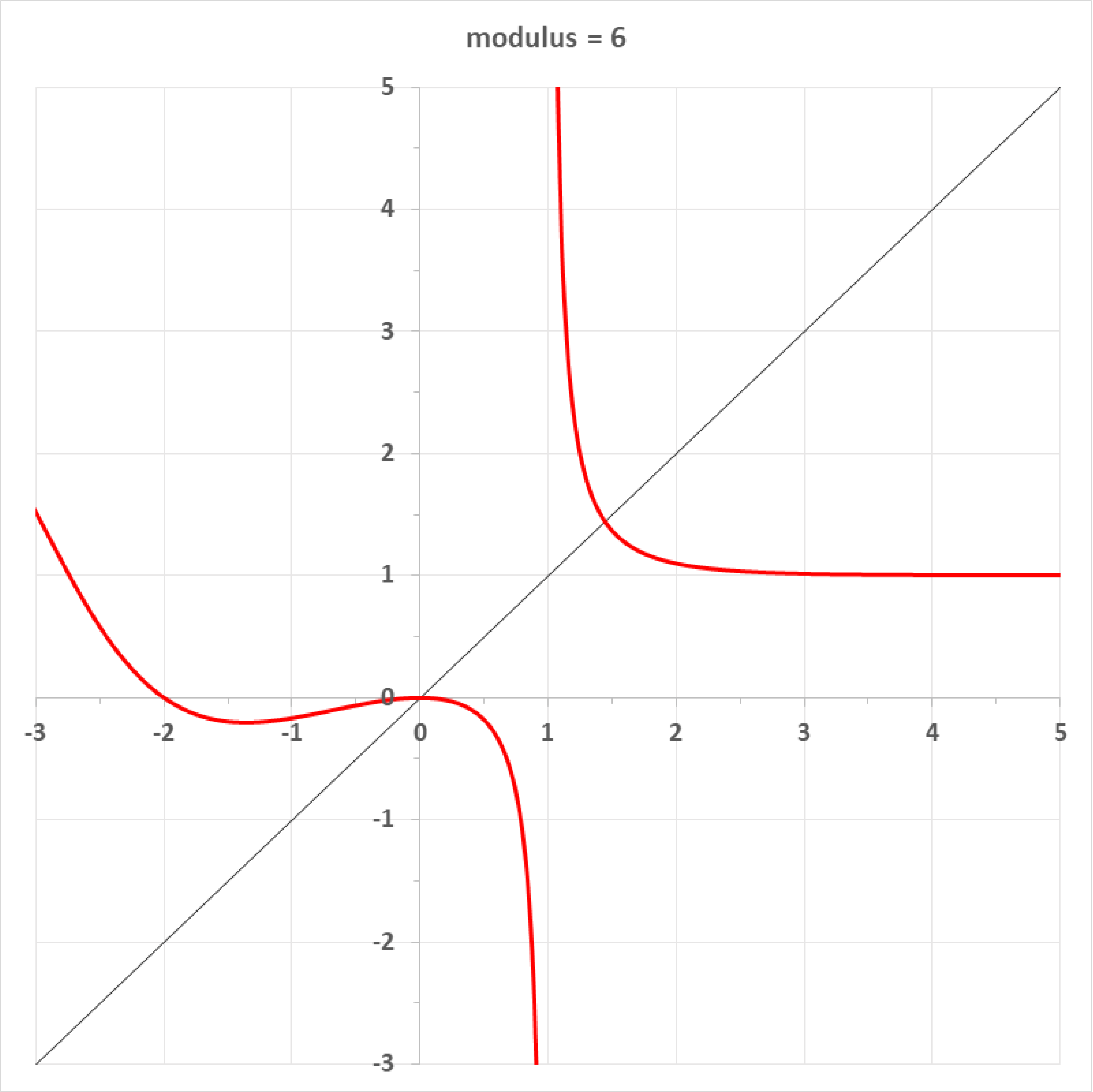
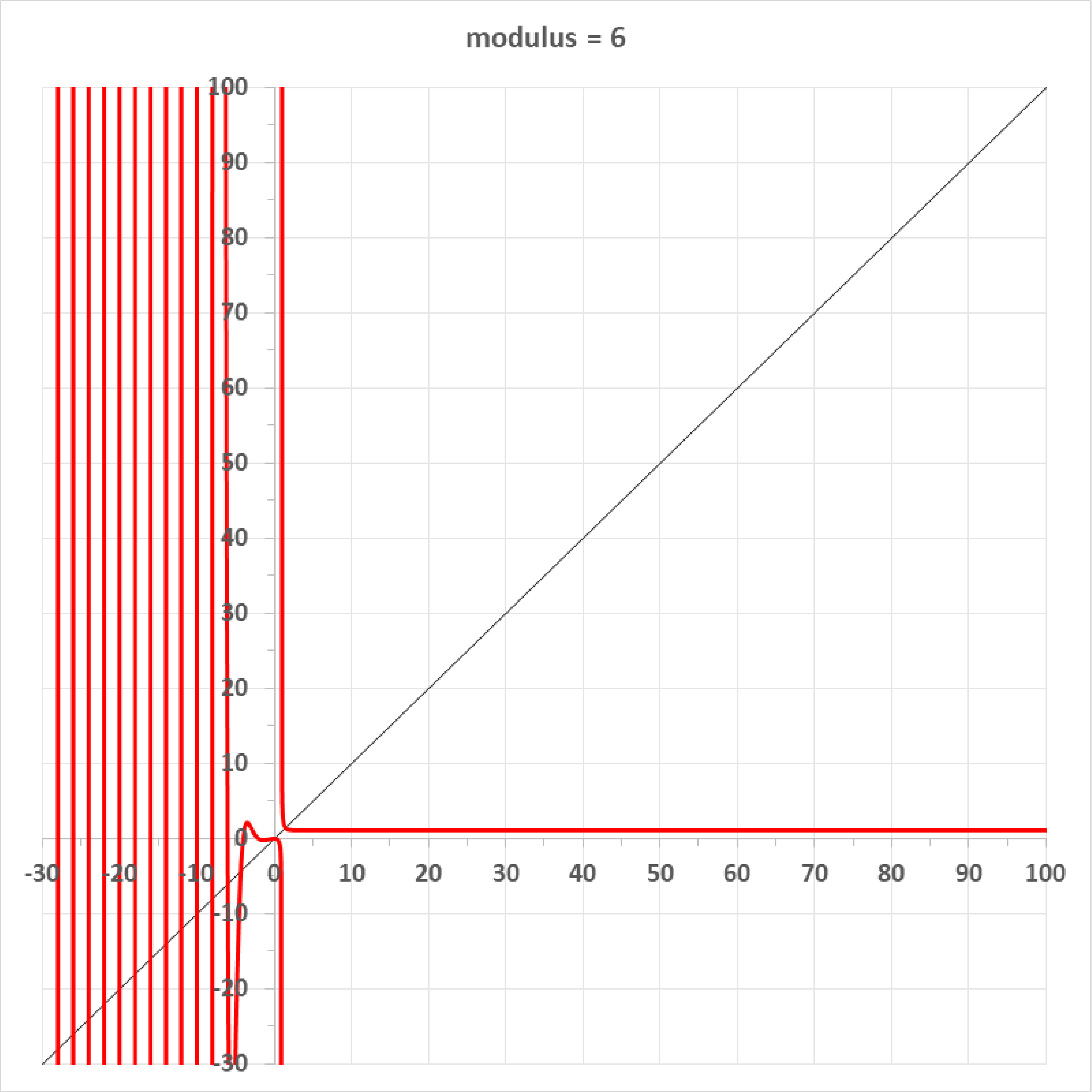
Dirichlet L function: χ_1 mod 6

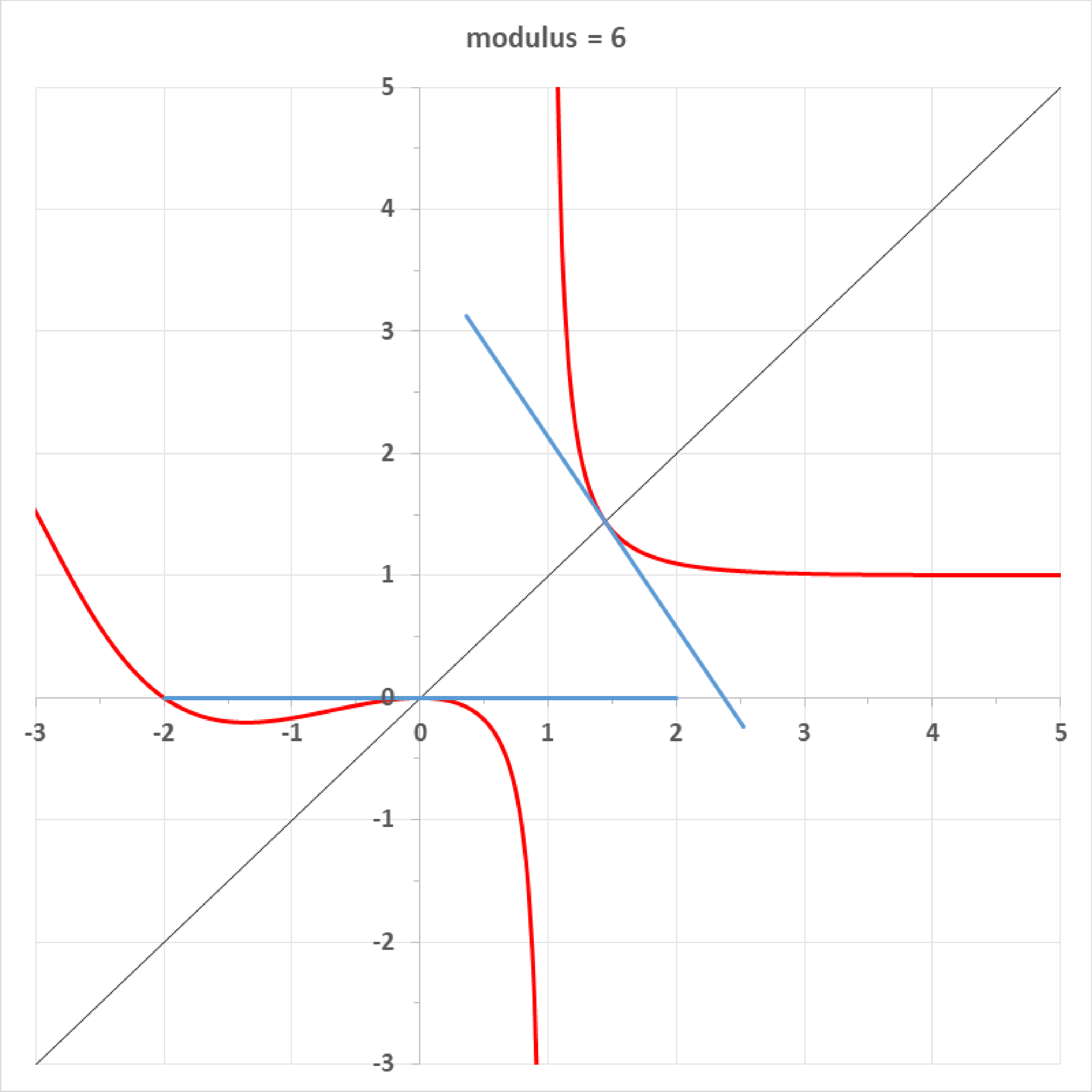




$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dirichlet L function: χ_1 mod 6





$-10 \leq \Re(s) \leq 6, -0.8 \leq \Im(s) \leq 26.5, 50ppu$

Dedekind η function

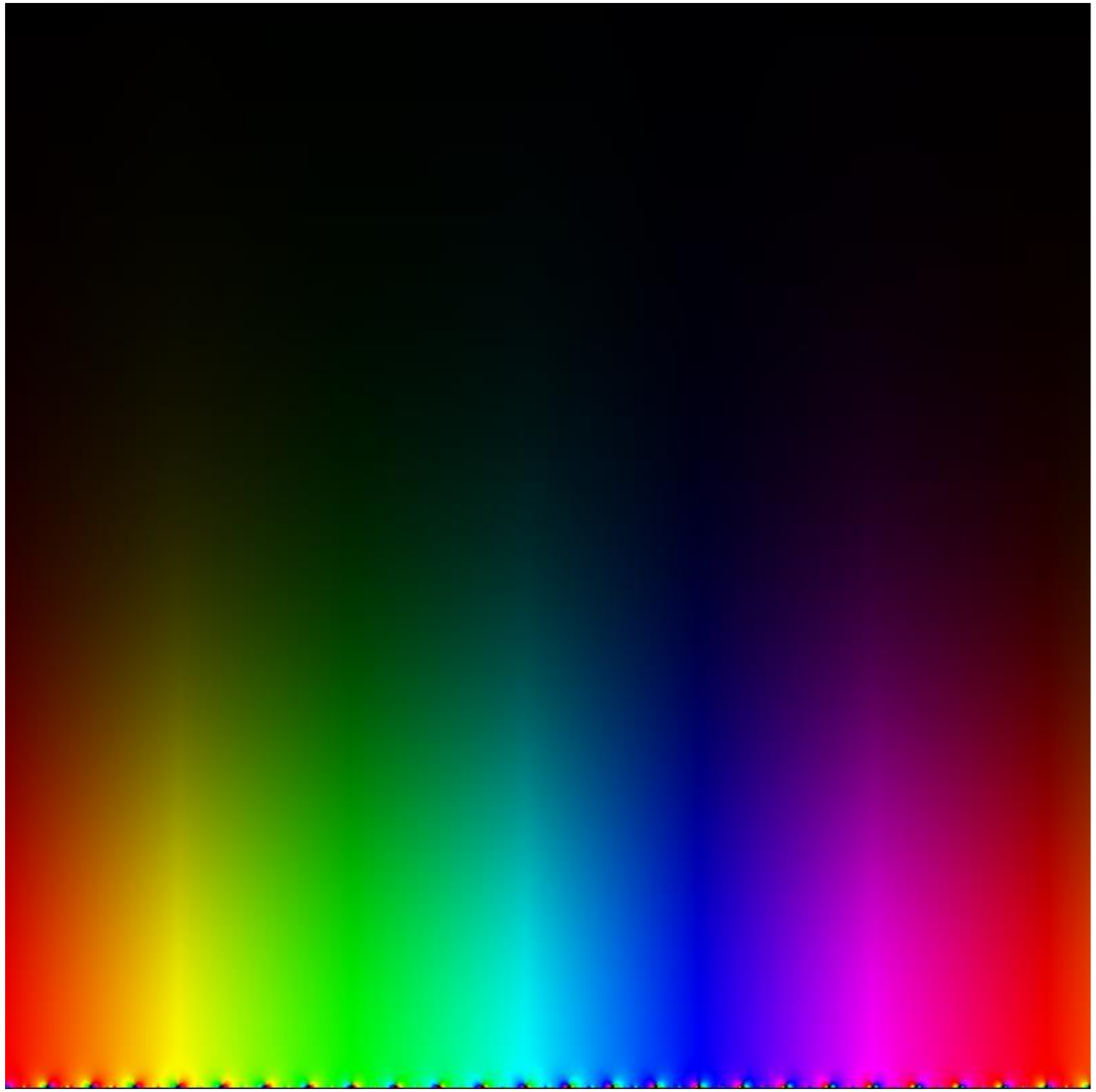
The Dedekind η function is a modular form of weight $\frac{1}{2}$ defined on the upper half plane $\mathbb{H}$ as:

$$\eta(\tau) := e^{2\pi i\tau/24} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i\tau n}\right) \qquad \Im(\tau) > 0 \tag{1}$$

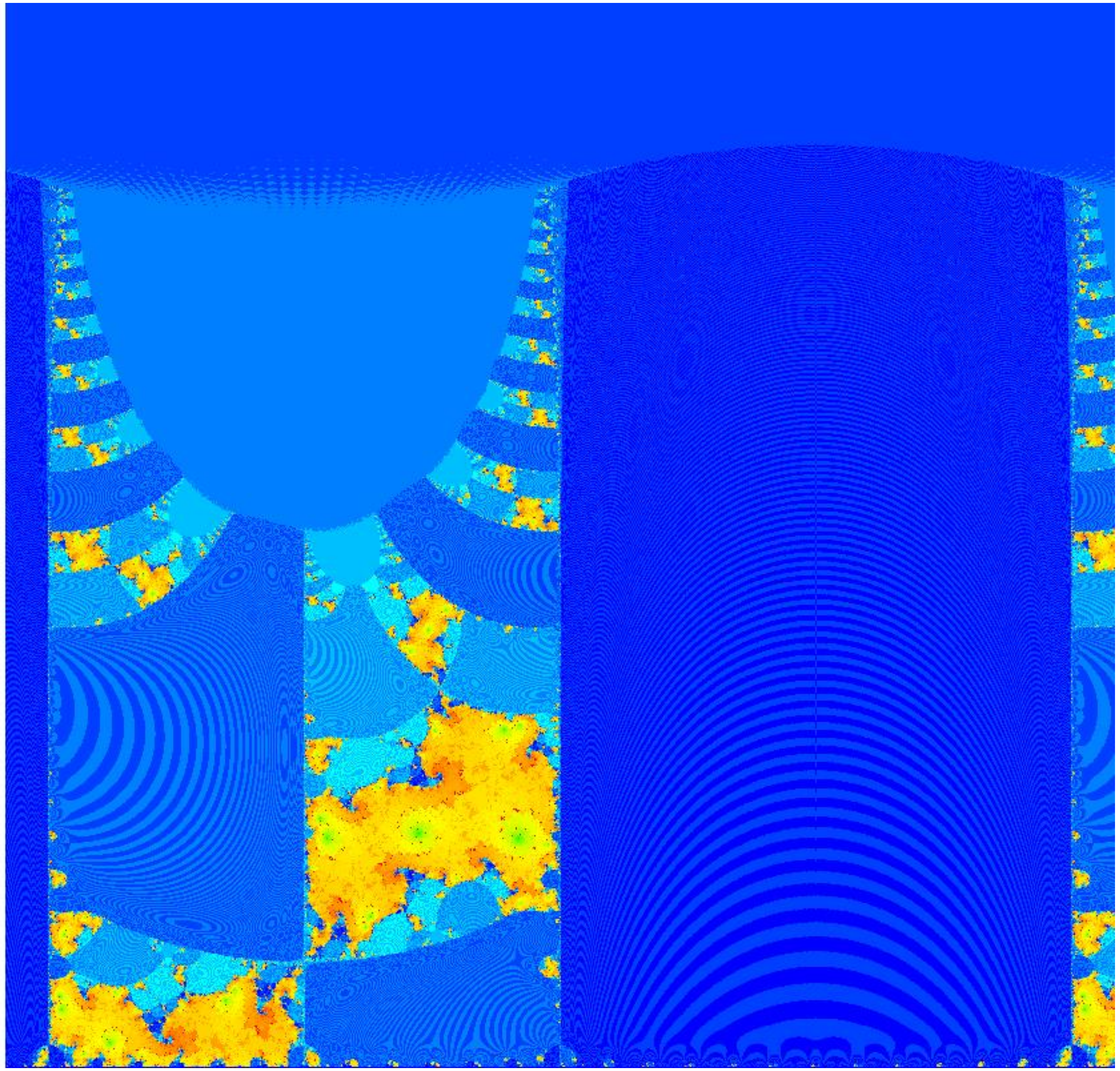
The quantity $q = e^{2\pi i\tau}$, known as the *nome*, is generally used in textbook definitions of the Dedekind η function:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \qquad \Im(\tau) > 0 \tag{2}$$

Dedekind η function

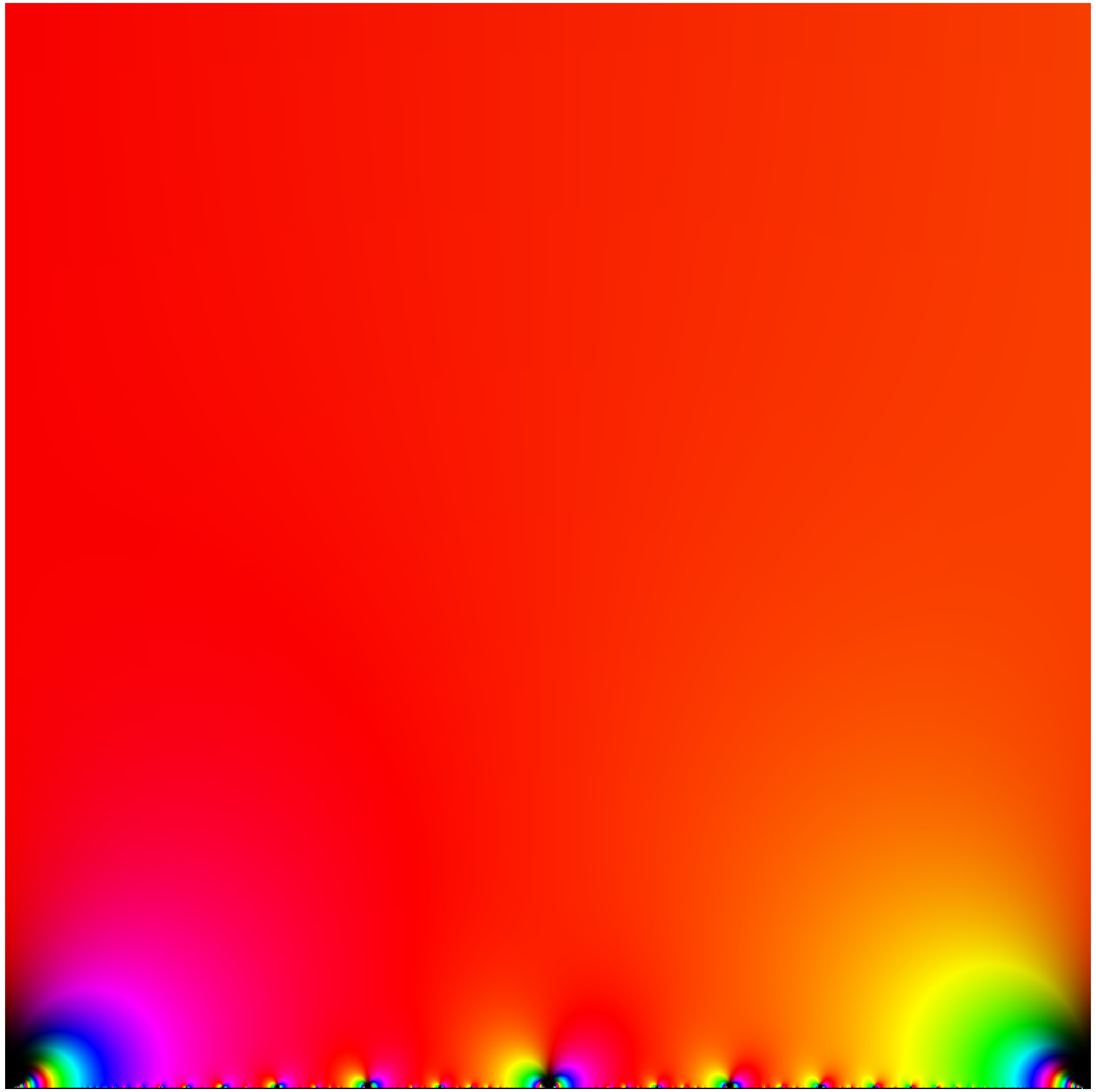


$-1 \leq \Re(s) \leq 25, 0 \leq \Im(s) \leq 25, 32ppu$

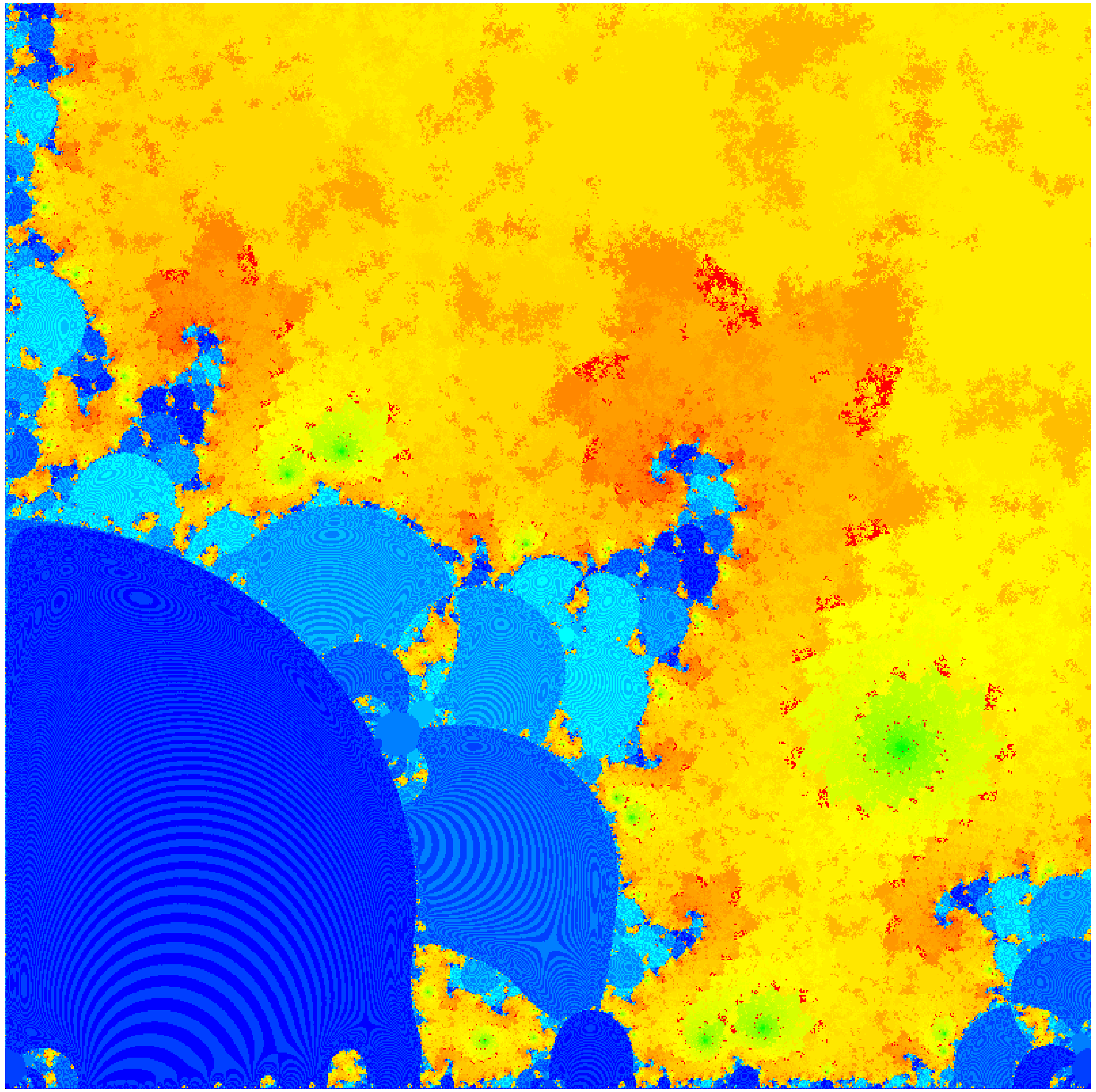


$-1 \leq \Re(s) \leq 25, 0 \leq \Im(s) \leq 25, 32ppu$

Dedekind η function

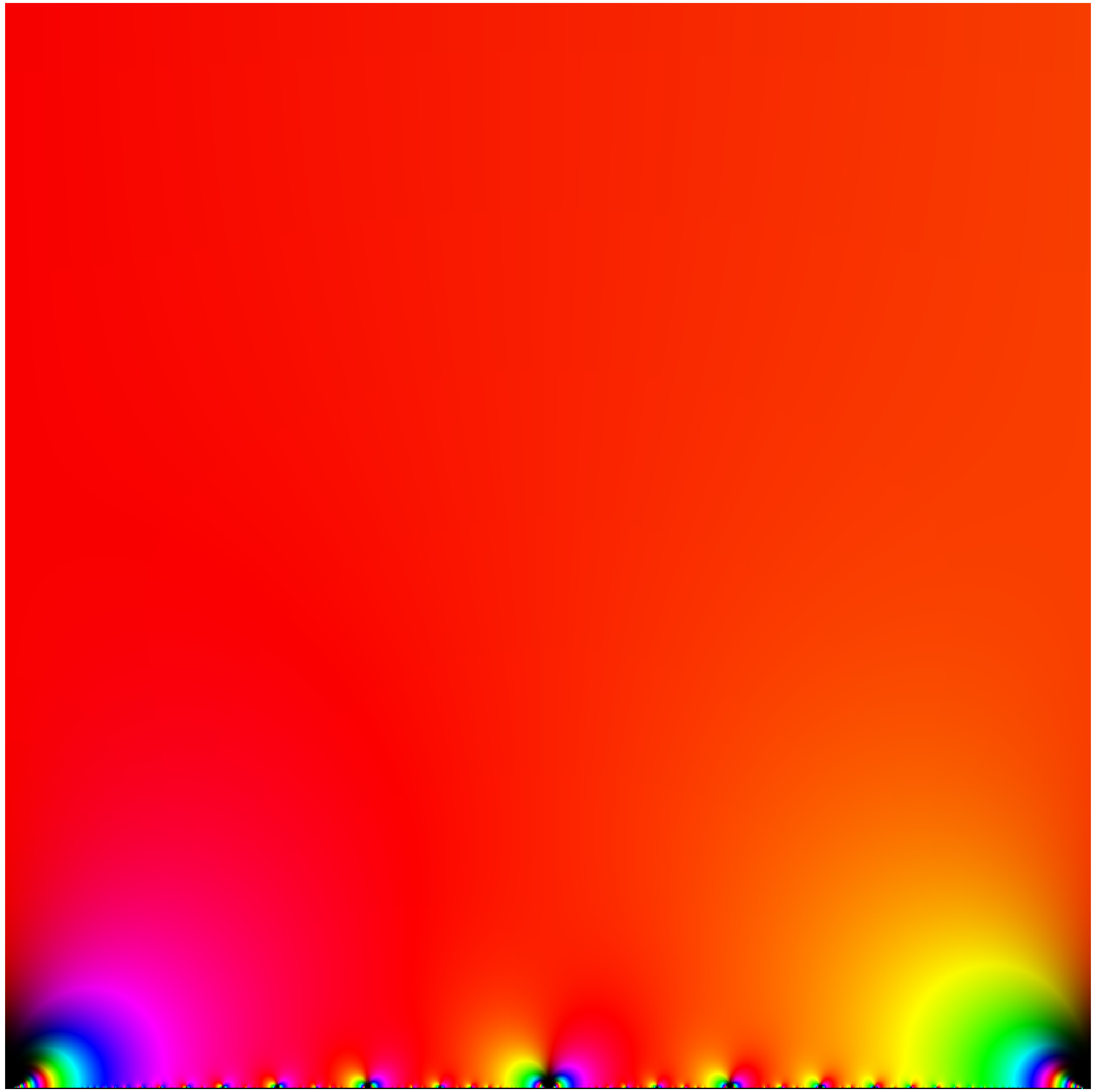


$0 \leq \Re(s) \leq 1, 0 \leq \Im(s) \leq 1, 800ppu$



$0 \leq \Re(s) \leq 1, 0 \leq \Im(s) \leq 1, 800ppu$

Dedekind η function



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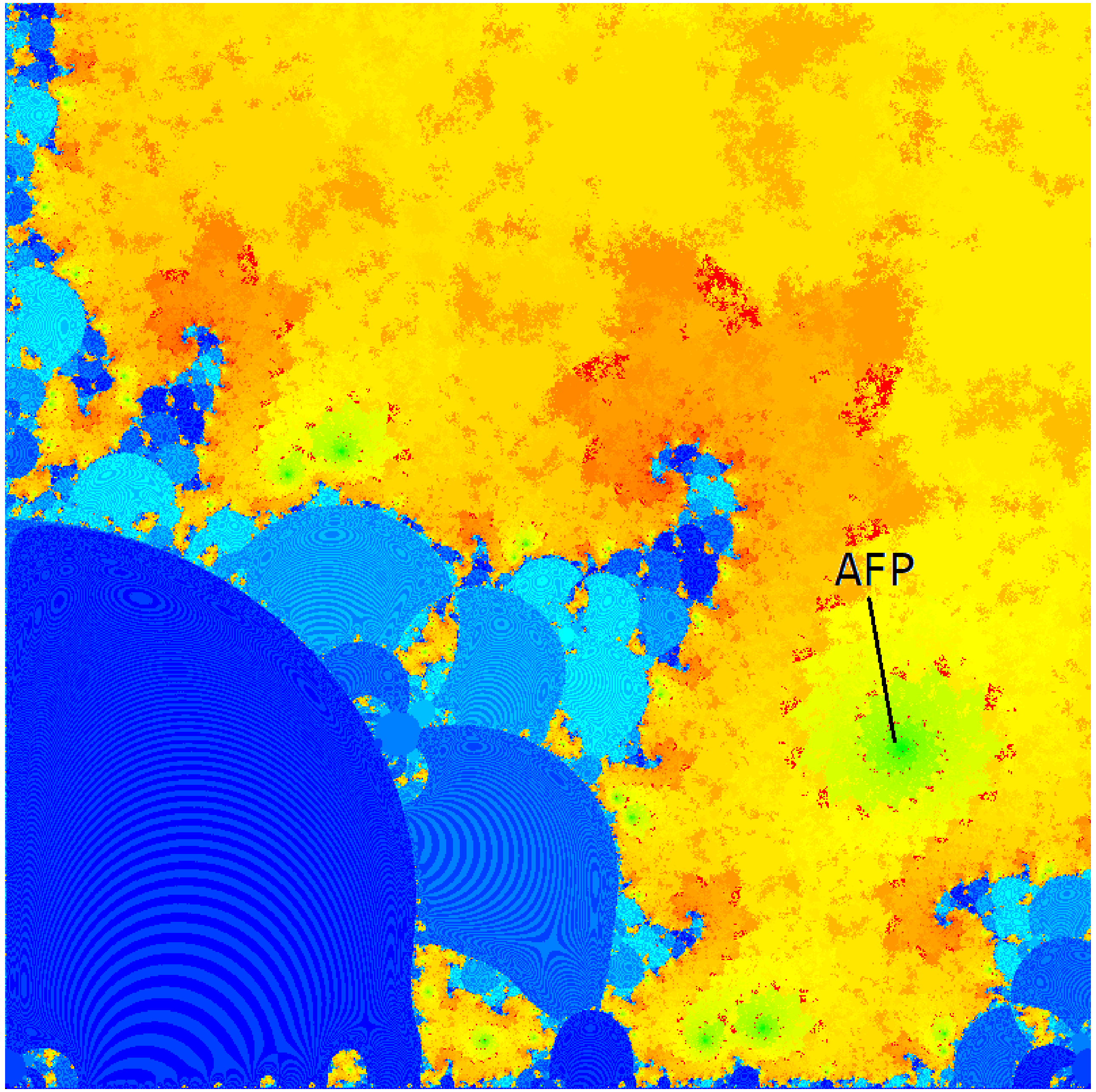
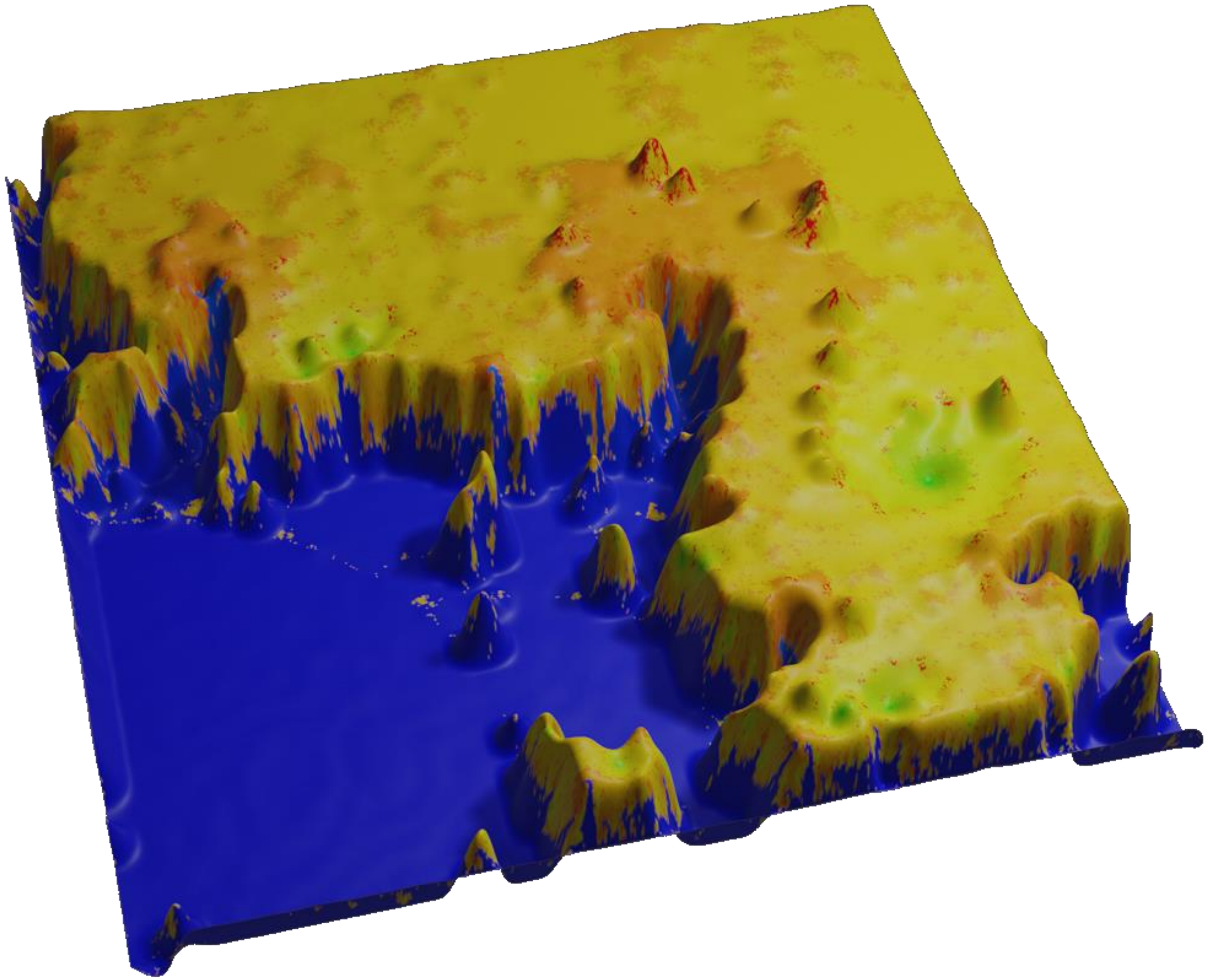
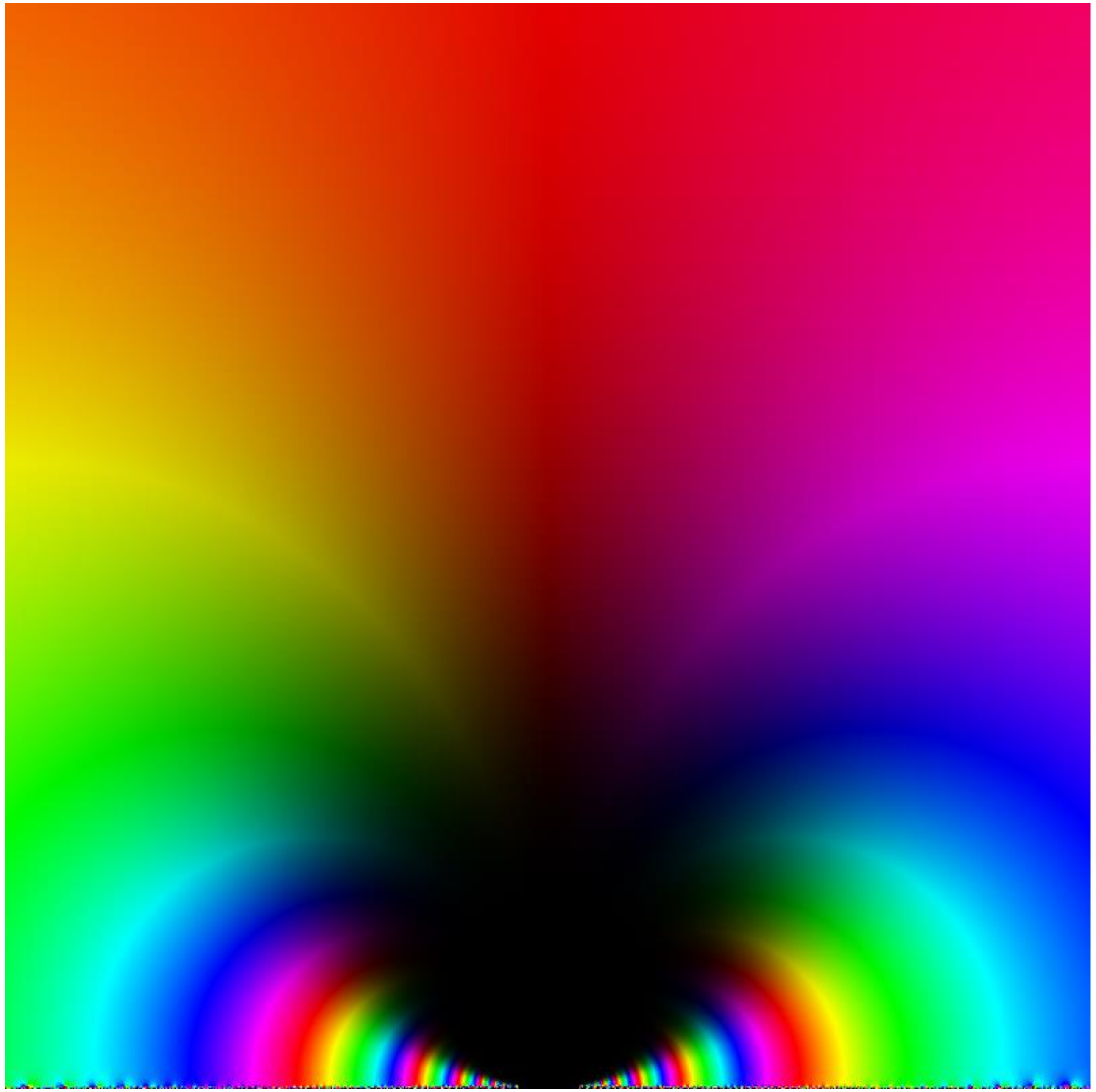


$0 \leq \Re(s) \leq 1, 0 \leq \Im(s) \leq 1, 800ppu$

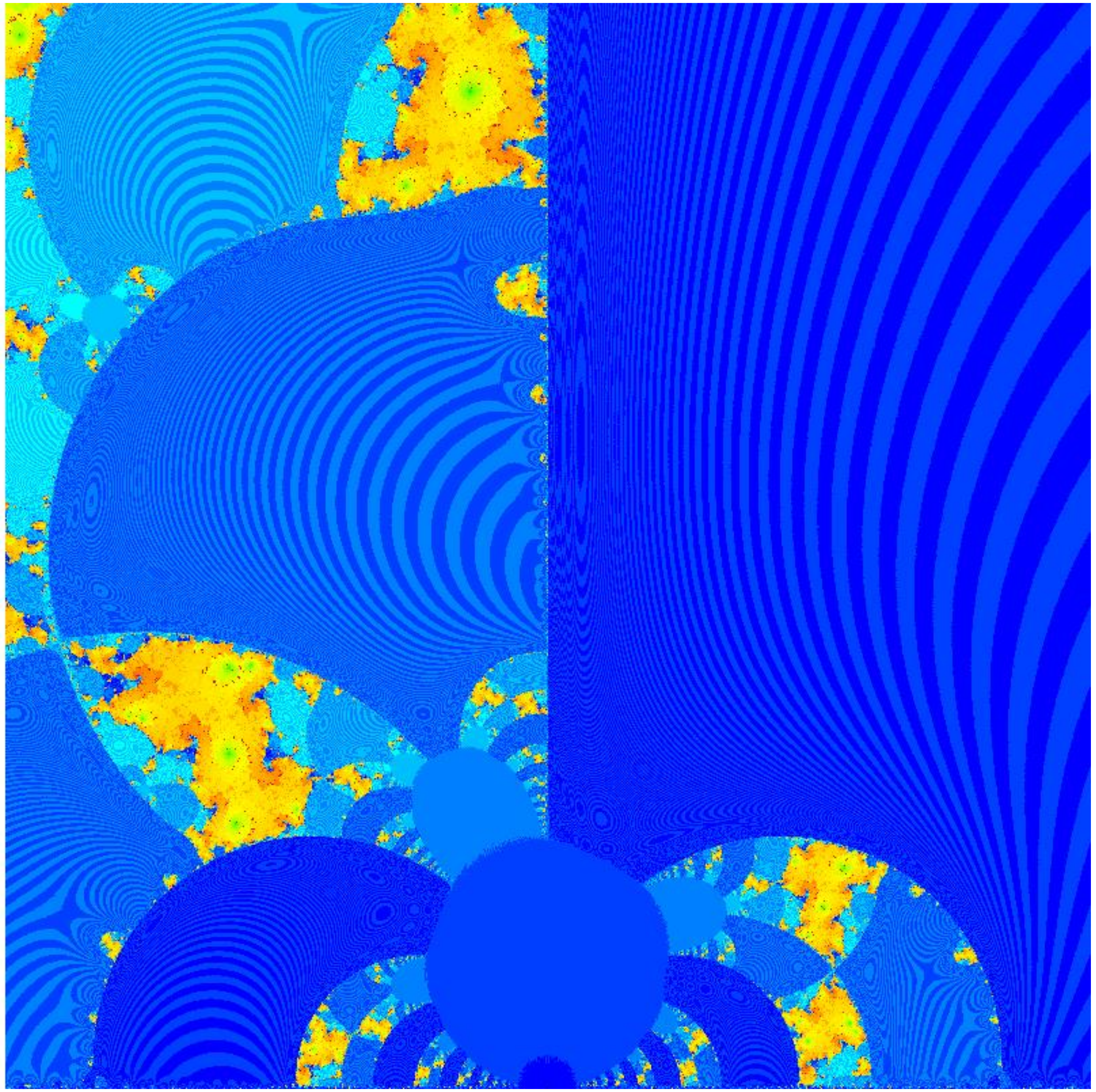
Dedekind η function



Dedekind η function



$-0.08 \leq \Re(s) \leq 0.08, 0 \leq \Im(s) \leq 0.16, 5000ppu$



$-0.08 \leq \Re(s) \leq 0.08, 0 \leq \Im(s) \leq 0.16, 5000ppu$

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